Preface

A special English language issue of the Hungarian Mathematical and Physical Journal for Secondary Schools (KöMaL) appears now for the second time. The first one, which was edited in Summer 1994, celebrated the centennial of founding the periodical.

This edition was made in honour of the 2nd European Mathematical Congress. The place of this outstanding event is Budapest. We would like to give a taste of the knowledge, interest and possibilities of gifted Hungarian and European high school students in mathematics and physics to the scientists visiting our country.

One of the satellite conferences of ECM2 is the Junior Mathematical Congress in Miskolc. In our special issue we publish some of the lectures sent to Miskolc. These papers (though most of the writers are 16–18 year-old students) lean in the direction of a high scientific level. One can see from these articles and abstracts the variety of themes which the students are interested in.

In the English edition we have collected all the mathematics and physics problems which were submitted in the journal in the last school year. That means 201 exercises and problems in mathematics, 97 in physics. Solutions can be read continuously in the numbers of KöMaL from 1996 to 1997 in Hungarian language. After the physics problems we carry an article written by two students for the International Conference of Young Scientists. This conference was held in Visegrád, in May, 1996.

If you find this journal interesting you may be also interested in the original Hungarian periodical. You can find some information on subscription and on the points contests of the journal on the last page.

Here I want to thank Péter Benczúr, Katalin Fried and Péter Körtesi for their valuable help in preparing this edition.

The editor

Budapest Welcomes ECM2

The János Bolyai Mathematical Society has been chosen by the European Mathematical Society to organize (under their auspices) the Second European Congress of Mathematics. The honour of getting the right to be the organizer of the Congress is certainly due partly to the high international reputation of Hungarian mathematics.

This high mathematical culture stems from the high level of education in elementary and secondary schools. KöMaL plays an exceptional role in this education. Most of the internationally accepted Hungarian mathematicians started their "career" by reading, studying and solving the exercises and problems of KöMaL. Some of the todays scientists can even thank their mathematical interest to KöMaL.

The motto of the Congress, "Unity of Mathematics", can be understood not only in a scientific sense but in a geographical sense as well: half of the approximately 1000 participants arrive from Central and Eastern Europe, the other half from Western Europe. A scientific congress of this size is a great opportunity for meeting people and for listening to distinguished colleagues. The plenary addresses serve the "Unity of Mathematics" by widely accessible lectures on the border of different fields or connecting different branches of mathematics.

There is one distinctive feature of the European congresses, the Round Tables, that increases the speciality of this event. Their main objective is to discuss the social aspects of mathematics in front of the general public. These conversations feature materials such as the role of mathematical games in education, the public image of mathematics, women in mathematics, and others.

We are looking forward to a successful meeting, and we are proud of being the host of this great event.

Antal Balog
Secretary of the Organizing Committee
of the Second European Congress of Mathematics

Junior Mathematical Congress '96

The Special English Language Issue of KöMaL with five articles and some abstracts in it sent to the Junior Mathematical Congress, is designed to encourage young participants to place one of their first published papers on the pages of a journal with more than a hundred years tradition.

The Junior Mathematical Congress '96 is one of the official satellite meetings of the 2nd European Congress of Mathematics, and it is aimed at bringing together the future mathematicians of Europe. We welcome young people interested in mathematics at this venue. The official languages of the congress are English and Hungarian.

Apart from attending lectures given by invited scholars and meeting famous European mathematicians, the participants may themselves give talks or exhibit posters. All active participants will be given the chance to present their results in subject oriented parallel sessions. In addition to the new ideas, the participants will become acquainted with yet unknown branches and applications of mathematics as well as educational softwares and logical games. The difference of age and mathematical background of the participants will be taken into consideration in organizing the lectures and the presentation of contributions at three different levels.

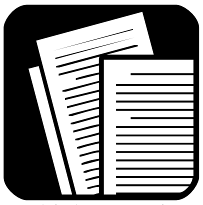
This meeting will also provide an excellent opportunity for the participants to make friends and to visit this part of Hungary. The welcoming party, various events and trips around Miskolc will complete the programme.

Those wishing to give a talk, present a poster or mathematical software should have submitted it in a written form. Our colleagues in the Scientific Committee had the difficult job of choosing some of the best papers to be published here as the result of this exciting experience. The European Mathematical Society has offered a grant for a few of the best papers to be published in this special issue, but we consider that each of the active participants deserves due recognition.

Let this short introduction be an opportunity to express my thanks for all participants and all my colleagues involved in the congress.

Péter Körtesi
Miskolc University
Chairman of the
Junior Mathematical Congress '96 Miskolc

Abstracts for JMC–96



On the next five pages we would like to touch some of the fields which came into prominence among the submitted papers. As we cannot carry all the articles, here you can find some abstracts of interesting papers, softwares and posters.

The Bolyai Geometry

Abstract

One of the first establishers of the non-Euclidean geometry was the Hungarian scientist János Bolyai. The aim of this talk is to explain at a high-school level the contribution of the greatest Hungarian mathematician to the development of geometry, and how Euclidean geometry splits into several non-Euclidean geometries. In order to understand Bolyai's contribution we need to review some of the history of the problem.

At the beginning geometry was a collection of results with no logical structure at all. From the work of the ancient Greek mathematicians we develop the notion that some of the geometric statements, the theorems, require proofs. A major step was the creation of the "Elements" of Euclid, summing up all the important results that the geometry of his age had achieved. He put the axioms, definitions and theorems into a logical order, where each statement follows from the previous one. In other words, he created the first system of axioms for geometry.

His Postulate V., known today as the "axiom of parallels" (in the plane determined by a point and a line not incident with it, there is only one line not intersecting the given line, incident with the given point), was much discussed. It was not simple enough, and it appeared to be deducible from other axioms. In fact, this controversial statement is independent of the other axioms in the system, and so the 2000 years of attempts to

prove or disprove it from other axioms were vain. Moreover, if we replace it by its contradiction, we get a new geometry, known today as hyperbolic geometry. In this new geometry the sum of the angles of a triangle is less than π ; the area of a triangle will depend on the sum of its angles; the totality of points equidistant to a line in a given plane is no longer a straight line but a curve; two similar triangles will be congruent.

This is the main result of János Bolyai, creating a new world in geometry by his new insight. His method consisted of leaving the Axiom V. unchanged, but inventing a new definition for parallelism. The result is an absolute geometry containing the Euclidean geometry and a non-Euclidean geometry as well. This new definition for parallelism and the use of the "remaining" system of axioms after excluding the controversial axiom, are treated in the Appendix to his father's work "Tentamen". The geometry constructed from the system of "remaining axioms" is called the "absolute geometry". The first theorems using neither the axiom of parallels, nor its negation are called "absolute geometry theorems". Later on, Bolyai introduces the notions of paracycle and parasphere, their properties, and the geometry of the surface of the parasphere. This surface is important as a model for non-Euclidean geometry. Later in the last century it was revealed that hyperbolic geometry can be modelled in Euclidean geometry, so as a consequence, the non-Euclidean geometry is consistent if the Euclidean system is.

There exist non-Euclidean geometries other than the Hyperbolic one. The important question is which one of them represents the real world surrounding us. The creation of non-Euclidean geometry was a revolution in mathematics which refused the view of mathematics being static. The belief that the system of axioms that Euclid created could not be changed was ended and it is an honour for us Hungarians, that this revolutionary change was initiated by a Hungarian mathematician, János Bolyai.

Anikó Csekes, Miklós Hosszu, Gergely Kardos,
László Mészáros, István Pataki, Lili Sitkei, Máté Uher
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Fascinating Cyclics Beyond the Pythagorean Theorem

Abstract

As an activity, my geometry class did a construction to graphically check the Pythagorean theorem. In the construction, four congruent cyclic quadrilaterals were formed out of the square constructed off the longer leg of a right triangle. These four quadrilaterals, arranged with the square constructed off the smaller leg of the right triangle, formed a square which, as expected, had an area equal to that of the square drawn off the hypotenuse.

We were then challenged to see if we could form the same cyclic quadrilateral by bisecting two interior angles of a triangle and their corresponding exterior angles. We found that to produce the same cyclic quadrilateral, we needed to use a right triangle for the second construction, but we also found a simple relationship between the right triangles needed for the two constructions. We found that the measure of one of the acute angles from the triangle in the second construction had to be twice the measure of the smallest angle of the triangle in the first construction if the quadrilaterals produced were to be similar. We also found the relationship between the sizes of the two triangles needed to make the quadrilaterals totally congruent.

Then came the biggest challenge. We were asked to see if we could form this same quadrilateral by bisecting the four interior angles of another quadrilateral. I found that the quadrilateral that I needed to bisect was a right trapezoid. Furthermore, I created a formula to find the dimensions needed for the trapezoid to get the desired cyclic quadrilateral. However, the way in which I did it is even more intriguing than the formula itself.

I have diagrams and animations from a computer program called Geometer's Sketchpad that show how I created the formula and discovered the relationship between the trapezoid and the right triangle from the original "graphic check" construction.

As with any great math problem, one question leads to another, and I am now exploring the relationships with other quadrilaterals besides that specific type of cyclic quadrilateral. This paper opens minds to the intrigue of looking for the inevitable relationships that form in math problems.

Timothy Stonehocker
Cornerstone Academy, Wheaton, U.S.A.

The Analysis of the Card-game "Black Jack"

Abstract

We have analysed the card-game Black Jack, which is the close relative of a Hungarian card-game "huszonegyes". The basis of our investigation was the collection of rules described by the Hungarian Casinos. We have neglected some smaller rules of minor importance. Since the game is played with 6 packs of French cards, we assumed that the probability of picking different cards is the same.

We have elaborated a recursive algorithm for calculating the probabilities of different sums of the picked cards at the end of a game. We propose an efficient strategy for the hand, which is based on the knowledge of the bank's first card and the sum of the value of the hand's first two cards.

We have written a computer program for the calculations. Using this method we can predict a 52.95% winning chance for the hand.

László Végh,
Fazekas Mihály Grammar School, Debrecen

Computing the Number of Fundamental Invariant of Binary Forms

Software

A software is presented that allows to count the number $F(n)$ of irreducible solutions of the equation

$$x_1+2x_2+\ldots+(n-1)x_{n-1}\equiv 0 \pmod{n}.$$

A table of values of $F(n)$ for $n\leqslant 64$ is calculated and some properties of the function $F(n)$ are observed.

V. Tsiskaridze
Tbilisi State University, Republic of Georgia

Do You Want to Play "Points"?

Software

We consider a computer program of a logical game very popular among belarussian students.

We need a sheet of cross lined paper and pens of different colours for this game. Each of the two players points a node per turn. The goal of each player is to enclose some points of the opponent by a closed polygonal line vertices of which being consecutive points of the same colour. The game is over when the paper is fulfilled. The player who surrounded more points than the partner is the winner.

Our program plays a role of one of the partners, and provides some additional auxiliary functions for playing the game.

Vladimir Samodurov
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Sundials and Spheric Trigonometry

Poster

The poster is about spheric trigonometry. It demonstrates and proves the most important theorems of this topic. Finally, it shows how one can calculate dials for sundials depending on its position on earth and the deviation of its plain. The poster shows some examples illustrating this.

Christian Saalfrank, Rico Gutschmidt
Hermann-von-Helmholtz-Gymnasium, Potsdam, Germany

'Taxi-geometric' and 'Motion-geometric' Studies

Poster

1. 'Taxi-geometric' studies

The starting problem was heard from Károly Kővári, a secondary school teacher (Budapest) in Nyíregyháza 20 years ago:

Imagine a square-lattice as a map of a city where the lines are the streets. What is the set of points the distance of which is n from a corner point (n is a given positive integer / real)?

What is the set of points which are in the same distance from two given corner points if the distance has to be integer / real?

(i) Studies of the second problem on a square grid map, with some generalisations.

(ii) Studies of equi-distance points to various shapes on a square grid map.

(iii) Studies of the second problem on regular triangular and hexagonal grid maps.

2. 'Motion-geometric' studies

The starting problem was heard from professor Ferenc Kárteszi in Nyíregyháza 20 years ago:

How many times will a coin turn around its own center while it is turning around a second coin of the same size?

(i) Construction of the original problem with proof and of the path of a point.

(ii) Construction for circles with $r=1/2$, $r=1/3$ and $r=1/4$ turning around a circle with $R=1$. The path of point A is also shown.

Generalisation: For $r=1/n$, the number of turns around its own center is $n+1$.

(iii) Construction for circles with $r=1/2$, $r=1/3$ and $r=1/4$ turning within a circle with $R=1$. The path of point A is also shown.

Generalisation: For $r=1/n$, the number of turns around its own center is $n-1$.

Gábor Szalontai, Rita Szalontai,
Krúdy Gyula Grammar School
and the pupil-team *of the*
Apáczai Csere János Practice School, Nyíregyháza

Some Topics in Many-dimensional Geometry

Poster

One may begin the study of many-dimensional Euclidean spaces with the consideration of geometry of some many-dimensional regular polyhedrons. As it is well-known ([1], Chapter 12), there are only three types of regular polyhedrons in the Euclidean space E_n for $n>4$: cubes, cocubes, and simplexes. The standard n-dimensional cube Cub_n, cocube Coc_n in $\mathbb{R}^n$, and the standard simplex Simp_n in $\mathbb{R}^{n+1}$ are given as follows:

$$\mathrm{Cub}_n=\{(x_1,\dots,x_n), 0\leqslant x_i\leqslant 1, \forall i=1,\dots,n\},\ \mathrm{Coc}_n=\left\{(x_1,\dots,x_n), \sum_{i=1}^{n}|x_i|\leqslant 1,\right\},$$

$$\mathrm{Simp}_n=\left\{(x_1,\dots,x_{n+1}), \sum_{i=1}^{n+1}x_i=1, \quad x_i\geqslant 0, \forall i=1,\dots,n+1\right\}.$$

In the present talk we construct a representation of the surface of cube Cub_n as an $(n-1)$-dimensional polyhedron in $\mathbb{R}^{n-1}$. This representation enables us to find the shortest way along the surface of Cub_n connecting two given surface points of Cub_n. For example, the distance along the surface between two opposite vertices of Cub_n is equal to $\sqrt{n+2}$.

The next topic is the description of intersections of Cub_n with hyperplanes orthogonal to its diagonal. All such intersections can be obtained from an $(n-1)$-dimensional simplex by cutting off its vertices. In particular, the 3-dimensional hyperplane $\{x^1+x^2+x^3+x^4=2\}$ intersects Cub_4 by an octahedron (three-dimensional cocube).

Nonorientable two-dimensional manifolds can be embedded in many-dimensional spaces without self-intersections. We present the following embeddings of the infinite Möbius band and the Klein bottle (see [2]) in E_4 and E_5, respectively:

$$\vec{x}=\vec{e}_1\cos u^1+\vec{e}_2\sin u^1+\left(\vec{e}_3\sin\frac{u^1}{2}+\vec{e}_4\cos\frac{u^1}{2}\right)u^2, \qquad u^1\in[0,2\pi],\ u^2\in(-\infty,+\infty),$$

$$\vec{x}=\vec{e}_1\cos u^1+\vec{e}_2\sin u^1+\left(\vec{e}_3\sin\frac{u^1}{2}+\vec{e}_4\cos\frac{u^1}{2}\right)\sin u^2+\vec{e}_3\cos u^2,\qquad u^1,\ u^2\in[0,2\pi].$$

References

[1] M. Berger, *Géométrie. Vol. 1 (Russian transl.)*, Mir, Moscow, 1984.

[2] C. Kosniovski, *A First Course in Algebraic Topology (Russian transl.)*, Mir, Moscow, 1983.

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János Bolyai the First Hungarian Number Theorist

János Bolyai (1802–1860) belongs to the greatest creators of science due to his achievements in the field of geometry. He is the discoverer of absolute geometry and one of those who created the non-Euclidean geometry.

In addition to his geometrical researches, János Bolyai was dealing successfully with several number theoretical problems, too. His results in this field are not only completely unknown in the history of mathematics, but moreover, all the authors of monographs which present Bolyai's life and work agree with the opinion that he has never been attracted to number theoretic problems. However, if we search and decipher carefully Bolyai's manuscripts — 13.000 pages, safely preserved in the famous Teleki–Bolyai Library [1] at Tîrgu–Mureş — we can become convinced on the opposite of this. The manuscripts certify that János Bolyai has been actively interested in number theoretic problems, too.

Here are some of the number theoretic results of Bolyai which I was able to decipher from manuscripts in the last period:

1. Let us begin with the presentation of János Bolyai's researches related to the "little" Fermat theorem. This theorem states the following: if p is prime and it does not divide a, then

$$a^{p-1}\equiv 1 \pmod{p}, \tag{1}$$

which means that $a^{p-1}-1$ is divisible by p. For instance 2^6-1 is divisible by 7, $7^{18}-1$ is divisible by 19, while 2^8-1 is not divisible by 9.

Being asked by his father, János Bolyai began to deal with the converse of Fermat's theorem, namely with the question whether it follows from (1) that p is prime or not.

Bolyai was one of those who showed that for some integers a, there can be found such composite numbers p for which (1) holds. We meet such instances on several pages of his manuscripts, for example the congruences

$$2^{340}\equiv 1 \pmod{341=11\cdot 31},\qquad 4^{14}\equiv 1 \pmod{15},\qquad 5^3\equiv 1 \pmod{4}$$

and

$$2^{2^{32}} \equiv 1 \pmod{2^{32}+1}.$$

He found the first congruence in an interesting way. He showed that if

$$\frac{a^{p-1}-1}{q} \quad \text{and} \quad \frac{a^{q-1}-1}{p}$$

are integers, then

$$a^{pq-1} \equiv 1 \pmod{pq}.$$

Next he put $a=2$ and after checking some primes, he guessed $p=11$ and $q=31$ [2].

Unfortunately, these results never became known by the scientific community. So it could have happened that 40 years later J. H. Jeans rediscovered this result, which is known today as Jeans theorem.

2. We can find notes about the "two squares theorem" on many pages of the manuscripts. The history of mathematics credits this theorem to Fermat, even though it was asserted by Girard earlier. Some people call it Fermat's Christmas Theorem. It says that any prime p of the form $4k+1$ is a sum of two squares, uniquely defined by p. For instance we have:

$$5=1^2+2^2, \quad 41=4^2+5^2, \quad 73=3^2+8^2.$$

It was Euler in 1754 who first proved this theorem, and, of course, it appears in Gauss's Disquisitiones Arithmeticae (1801). This book was known by János Bolyai, too. János's father — Farkas Bolyai — who read Euler's proof in the Teleki Library, found it too long (indeed, it extends on over 55 pages) and sophisticated. This was why he encouraged his son to look for a simpler proof of this theorem. Soon János Bolyai, answering his father in a letter, sent him as many as three different proofs, all being written on just two pages!

It is a surprise that his proofs use complex numbers. He applied a theorem — found in Disquisitiones Arithmeticae — which says that if p is a prime of the form $4k+1$, then exists an integer x, such that

$$\frac{x^2+1}{p}$$

is integer, too. He wrote this fraction in the form

$$\frac{(x+i)(x-i)}{p},$$

and then, using the properties of complex integers, he was led to $p=a^2+b^2$.

We mention that it was only G. Eisenstein who proved the two squares theorem before Bolyai, also by using complex numbers (1844), but his proof — which of course was not known by Bolyai — differs completely from that of János Bolyai.

3. It could sound surprising but I also found magic squares in Bolyai's manuscripts. The page of the manuscripts on which a magic square can be found, however, seems to be just the concluding part of a longer sequence of ideas. I have not yet found the supposed beginning of it. The magic square of János Bolyai is the following:

This exhibits the property that the sum of numbers in each row, column and each of the two diagonals is the same integer. It is remarkable that Bolyai built up this magic square in a general form, in which we get different kinds of magic squares when giving different values to the variables.

Until now we had the opinion that Hungarian number theoretic researches begun late and sporadically only in the last decades of the 19th century [3]. All the above, and also other works of János Bolyai show that he gained important results in the field of number theory several years earlier.

References

[1] János Bolyai, *Manuscripts*, Teleki — Bolyai Library, Tîrgu–Mureş.

[2] Kiss, E., *Fermat's Theorem in János Bolyai's Manuscripts*, Mathematica Pannonica, 6 , (1995), pp. 237–242.

[3] Szénássy, B., *A magyarországi matematika története*, Akadémiai Kiadó, Budapest, 1970.

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On Some Generalisations of the Fibonacci Sequence

In this article I shall describe some results obtained at our workshop for secondary school pupils interested in mathematics, in Erlangen, Germany, organised and guided by **Prof. Dr. J. Cofman** of the Friedrich-Alexander Universität Erlangen-Nürnberg. The members of the group were **Agnes Föglein**, **Anna Föglein**, **Andreas Giese**, **Eva Hiemer**, *Peter Kiefer*, **Michael Lorenz**, **Steffen Weiss**, **Edith Werner**, **Teresa Werner**, **Julianna Zsido**.

Christian Korn, a first year student of mathematics, acted as a tutor.

We started with the study of the Fibonacci numbers. These were first mentioned in the famous book 'Liber abaci' written in the 13-th century by Leonardo of Pisa, nicknamed Fibonacci. (see e.g. [2]).

The Fibonacci numbers form the sequence

$$1, \quad 1, \quad 2, \quad 3, \quad 5, \quad 8, \quad \dots$$

Let us denote the n-th term of the sequence by f_n for $n=0, 1, 2, \dots$.

The terms of the sequence satisfy the recurrence relation

$$f_n = f_{n-1} + f_{n-2} \qquad \text{for } n \geqslant 2$$

with the initial conditions $f_0 = f_1 = 1$.

The Fibonacci numbers appear in many branches of mathematics. There are also many puzzles and problems involving these numbers. We looked at the following problems:

Problem 1. A rectangle $R_{1,n}$ of size $1 \times n$, where $n = 1, 2, 3, \dots$ should be covered without gaps and overlapping with two kinds of tiles: square tiles of size 1×1 and rectangular tiles of size 1×2.

The question is that how many ways this can be done?

Solution. Let g_n denote the number of ways for covering $R_{1,n}$ in the way described above. It is easy to show that g_n satisfies the same recurrence relations as f_n :

$$g_n = g_{n-1} + g_{n-2} \qquad \text{for } n > 2 \tag{1}$$

with the initial conditions

$$g_1 = 1 \qquad \text{and} \qquad g_2 = 2. \tag{2}$$

(1) and (2) are illustrated in Fig. 1.

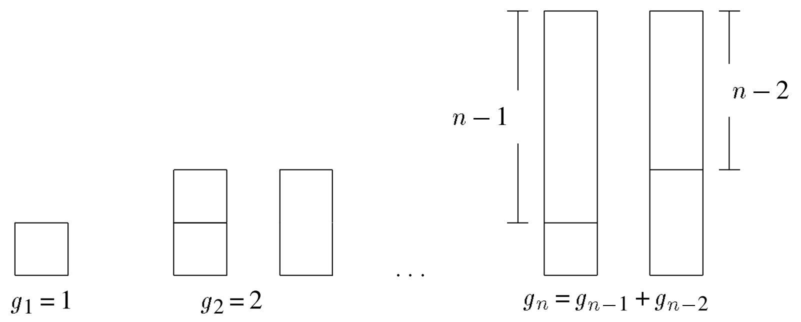

Figure 1

Thus $g_n = f_n$ for $n \geqslant 1$.

Our next aim was to study some generalisations of problem 1. One possibility was to keep the width of the rectangle the same while extending the set of the sizes of the tiles.

Problem 2. Let t_n denote the number of ways for covering $R_{1,n}$ without gaps and overlapping with tiles of size 1×1, 1×2, 1×3. Find a recurrence formula for t_n.

Solution. Fig. 2 indicates that t_n satisfies the recurrence relation

$$t_n = t_{n-1} + t_{n-2} + t_{n-3} \quad \text{for } n > 3 \tag{3}$$

with initial conditions $t_1 = 1$, $t_2 = 2$, $t_3 = 4$.

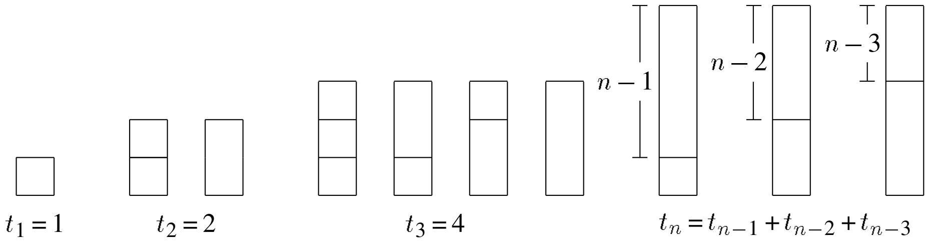

Figure 2

Having considered various tilings of $R_{1,n}$, Teresa suggested changing the width of the rectangle. The situation became much more complicated since some of the rectangular tiles could be placed in two different ways — horizontally or vertically.

Therefore we first focused our attention on rectangles $R_{2,n}$ of size $2 \times n$ for $n = 1$, 2, ... which had to be covered by tiles of size $1 \times 1, 1 \times 2$ only:

Problem 3. Let a_n denote the number of ways for covering $R_{2,n}$ without gaps and overlapping with tiles of size $1 \times 1, 1 \times 2$. Find a recurrence relation for a_n.

Solution. $R_{2,n}$ can be partitioned in $2 \times n$ unit squares. Let L denote the unit square in the left bottom corner of $R_{2,n}$ (Fig. 3a). L can be covered in three different ways with tiles of size 1×1 and 1×2. Therefore we can distinguish three types of tilings of $R_{2,n}$:

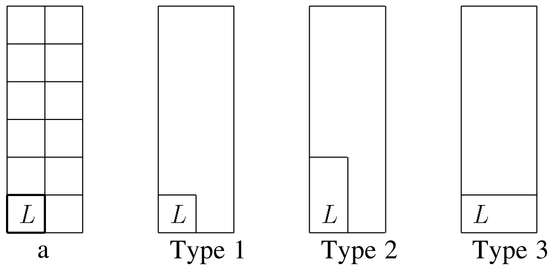

Figure 3

Type 1: Tilings in which L is covered by a tile of size 1×1.

Type 2: Tilings in which L is covered by the lower half of a tile of size 1×2 (the tile is in vertical position).

Type 3: Tilings in which L is covered by the left half of a tile of size 1×2 (the tile is standing horizontally).

Let b_n denote the number of all coverings of $R_{2,n}$ which are of Type 1, and c_n the number of all coverings of $R_{2,n}$ which are of Type 2. We can easily get three recurrence formulas: one for a_n, another for b_n and a third one for c_n :

$$a_n = b_n + c_n + a_{n-1} \tag{4}$$

$$b_n = a_{n-1} + b_{n-1} \tag{5}$$

$$c_n = b_{n-1} + a_{n-2} \tag{6}$$

Substituting (6) into (4) we obtain

$$a_n = b_n + b_{n-1} + a_{n-2} + a_{n-1}. \tag{7}$$

Combining (5) with (7) yields

$$a_n = 2b_n + a_{n-2} \tag{8}$$

that is

$$b_n = \frac{1}{2}(a_n - a_{n-2}) \tag{9}$$

It follows from (9) and (7) that

$$a_n = 3a_{n-1} + a_{n-2} + a_{n-3} \tag{10}$$

(10) is a recurrence formula expressing a_n in terms of three previous terms of the sequence, for all $n > 3$. The initial conditions are $a_1 = 2$, $a_2 = 7$ and $a_3 = 22$, which can be verified easily.

Recurrence formulas for the number of coverings of $R_{3,n}$ or $R_{4,n}$ with tiles of size 1×1 and 1×2 can be established by applying the same method. The procedure, however, becomes longer and much more tedious.

Eva and Christian extended the study of tilings to rectangles $R_{m,n}$ of size $m \times n$ with tiles of size 1×1, 1×2, ..., $1 \times k$ $(k \in N)$. They obtained a recurrence formula for the number $f(m, n, k)$ of such tilings. Moreover, Christian established upper bounds for $f(m, n, k)$ for $k \in N$. (see [1])

References

[1] E. Hiemer, Ch. Korn: *Pflasterungen von Rechtecken mit Steinen vom Typ* $1 \times j$, to appear in 'Elemente der Mathematik'

[2] N. N. Vorobyov: *The Fibonacci numbers*, D.C. Heath and Company, Boston, 1966

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Chessboard-tilings by Rectangle Dominoes

Let us number the rows and columns of an $n \times m$ chessboard from 1 to n and m, respectively. (i, j) denotes the common field of the ith row and jth column.

Our aim is to find some equivalent conditions for tilability. All of the "sufficient" parts of the following statements are trivial, so we will prove only the "necessary" parts. Let us start with a problem from KöMaL.

Proposition 1. *An $n \times m$ board can be tiled by $k \times 1$ dominoes if and only if it can be done by translated copies of $k \times 1$ dominoes (i.e. $k \mid n$ or $k \mid m$).*

Proof. Let us suppose indirectly that the board can be tiled without $k \mid n$ or $k \mid m$. Let b and d denote the least positive remainder of n and m modulo k, respectively. We may assume that $b \leqslant d$. Then $1 \leqslant b \leqslant d \leqslant k-1$.

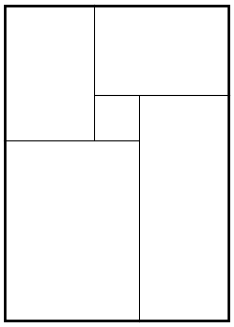
For every $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant m$, we write the least positive remainder of $i-j+1$ modulo k into the position (i, j). So one of the numbers $1, 2, \ldots, k$ is written into each field. Every $k \times 1$ domino covers exactly one set of the numbers $1, 2, \ldots, k$. Conclusively, the fact of tilability implies that the number of each of the numbers 1, 2, $\ldots$, k on the board is the same.

Divide the chessboard into sub-boards by drawing a horizontal and vertical line after every $(b+ik)$-th row and $(d+jk)$-th column, respectively ($i, j=0, 1, \ldots$). A side of all these sub-boards except one is divisible by k so they contain the same quantity of each of the numbers $1, 2, \ldots, k$. The exception is the one which is formed by the first b rows and the first d columns of the original board. There is a number 1 in (i, i) ($1 \leqslant i \leqslant b$) so because of $b \leqslant d$, every row of this sub-board contains number 1, but only once. However, the first row does not contain any 2's, and the others do it only once (just before 1) so this sub-board contains more 1's than 2's. Then the whole board contains more 1's than 2's, which is a contradiction.

Proposition 2. *An $n \times m$ chessboard can be tiled by $k \times h$ dominoes if and only if the chessboard or both of its sub-boards into which an appropriate straight line parallel with a side of the chessboard divides it, can be tiled by translated copies of $k \times h$ dominoes.*

Proof. It is necessary that the board can be tiled by $k \times 1$ respectively $1 \times h$ dominoes so $k \mid n$ or $k \mid m$ and $h \mid n$ or $h \mid m$. We are ready if $k \mid n$ and $h \mid m$ or $k \mid m$ and $h \mid n$. It is enough to deal with the case $k \mid n$ and $h \mid n$. (The case $k \mid m$ and $l \mid m$ can be done similarly.) It can be assumed that m is not divisible by either k or l.

Let us examine a horizontal side of the board. It is tiled by k and h long domino-edges, so the length of this side is $m = a \cdot k + b \cdot h$ (a; b are positive integers). Conclusively, the board can be divided by an appropriate vertical line into an $n \times ak$ and an $n \times bh$ sub-board, which can be tiled by translated copies of $k \times h$ dominoes.



Proposition 3. *We have cut out a field from an $n \times m$ chessboard and want $k \times h$ dominoes to tile it. Let us lengthen the sides of the field cut out with the same orientation in order to divide the board into 2, 3 or 4 rectangle sub-boards. The statement is that the whole board can be tiled if and only if all of the sub-boards can be tiled by $k \times h$ dominoes.*

Proof. As previously, first the case $h=1$ is examined. In this case the condition we gave for tilability is equivalent with the following ((a, b) is the field cut out):

$$n \equiv m \equiv 1 \pmod{k} \qquad \text{and} \qquad a \equiv b \equiv 1 \pmod{k}$$

or

$$n \equiv m \equiv -1 \pmod{k} \qquad \text{and} \qquad a \equiv b \equiv 0 \pmod{k}.$$

It is necessary that the number of fields is divisible by k, namely $n \cdot m \equiv 1 \pmod{k}$.

K☹MaL – 1996/6 ANGOL cianmagentayellowblack2002. j³nius 18. – 19:51 (14. lap)

Write the least positive remainder of j modulo k into (i, j). A $k \times 1$ domino covers one number k times or each of the numbers 1, 2, $\ldots$, k exactly once. Consequently, the number of each of the numbers 1, 2, $\ldots$, k on the board is the same modulo k. As k does not divide m, there are more columns containing 1 than k, exactly by one. Depending on the column containing the field cut out, the difference of the number of 1's and k's on the board is n, $n-1$ or $n+1$. Since this difference is divisible by k, it cannot be n. Hence the following two cases are possible:

I. The field cut out lies in a column containing 1. Then

$$n \equiv 1 \pmod{k} \qquad \text{and} \qquad b \equiv 1 \pmod{k}.$$

II. The field cut out lies in a column containing k. Then

$$n \equiv -1 \pmod{k} \qquad \text{and} \qquad b \equiv 0 \pmod{k}.$$

Similarly:

$$m \equiv 1 \pmod{k} \qquad \text{and} \qquad a \equiv 1 \pmod{k}$$

or

$$m \equiv -1 \pmod{k} \qquad \text{and} \qquad a \equiv 0 \pmod{k}.$$

The "mixed" cases (e.g. $m \equiv a \equiv 1 \pmod{k}$, $b \equiv 0 \pmod{k}$, $n \equiv -1 \pmod{k}$) are impossible if $k \geqslant 3$, since $n \cdot m \equiv 1 \pmod{k}$.

The case $k=2$ has remained. We have seen above that $n \equiv m \equiv 1 \pmod{2}$ is guaranteed, but the same parity of a and b is not. Fortunately, the traditional chessboard-colouring will finish our job. While n and m are odd, there is one more black field on the board than white. A 2×1 domino covers a white and a black field so the quantities of the black and white fields covered are the same. Conclusively, the field cut out is black which means that the parity of a and b is the same.

The case of a general $\underline{h}$. A vertical and a horizontal side of the chessboard can be chosen which do not contain the field cut out. These sides are tiled by k and h long domino-edges, so the length of these sides are $m = p \cdot k + f \cdot h$ and $n = c \cdot k + d \cdot h$, with p; f; c; d being positive integers. Let e denote the greatest common divisor of k and h. Then e divides m and n, conclusively $e^2 \mid mn$. On the other hand, $e^2 \mid kh$ and the $n \times m$ chessboard (except the field cut out) can be tiled by $k \times h$ dominoes, so $e^2 \mid n \cdot m - 1$, conclusively, $e^2 \mid 1$ namely $e = 1$, so k and h are coprime numbers.

Let us examine one of the sub-boards defined according to the statement. The rows and columns of the chessboard can be numbered in a way that this sub-board contains $(1, 1)$. According to the previous, this sub-board can be tiled with $1 \times h$ and $k \times 1$ dominoes, so a side of the board is divisible by k and a side is by h. If a side of it is divisible by k and the side orthogonal to this side is divisible by h, then the sub-board can be tiled by $k \times h$ dominoes. In the other case, a side of the sub-board is divisible by kh and we claim that the side orthogonal to it is $x \cdot k \cdot h \pm 1$ (x is a positive integer). We can prove it in the following way:

Let us assume that kh divides the number of the rows of the sub-board so $a \equiv 1 \pmod{k \cdot h}$ or $a \equiv 0 \pmod{k \cdot h}$. According to the above, $a \equiv b \pmod{k}$ and

$a \equiv b \pmod{h}$ so $a \equiv b \equiv 1 \pmod{k \cdot h}$ or $a \equiv b \equiv 0 \pmod{k} \cdot h$. Conclusively, the number of the columns is $x \cdot k \cdot h \pm 1$ (if not divisible by kh).

As k and h are coprime numbers, there exist such integers g, f $(-k \leqslant g \leqslant k;\ -h \leqslant f \leqslant h)$ that $f \cdot k - g \cdot h = 1$.

Conclusively, $x \cdot k \cdot h - 1 = g \cdot h + (x \cdot h - f) \cdot k$, $(x \cdot h - f \geqslant 0)$ and $x \cdot k \cdot h + 1 = (x \cdot k - g) \cdot h + f \cdot k$, $(x \cdot k - g \geqslant 0)$. According to Proposition 2, this sub-board can be tiled by $k \times h$ dominoes.

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Winning Strategies in Finite Games

This article will present some results about finding winning ways in so-called finite games. Of course, we need to clarify which games are included in the definition of "finite games".

In these games there are two players: Black and White. There is a starting position, and finitely many positions and moves are allowed by the rules. Black and White move alternately. The game ends within finitely many moves, and in the end one of the two players — who cannot move — is the loser, the other is the winner. Finite games have to come to an end in some moves, so there must not be any "loops" in the game. A loop means that a position can be reached twice during the game. Both players know the actual position exactly: they have complete information. Probability does not play any role in the game.

As one can see, this definition does not contain many games, for example tennis, because there are infinitely many positions; chess, because there can be never-ending loops and there can also be a draw; and many other games.

Our aim is to describe a method for finding the winning strategy in finite games. First we construct a directed graph with vertices representing the positions and directed edges representing the moves of the game. If you look at a vertex, it can have finitely many successors: the vertices to which an edge points from that particular vertex.

Obviously, there can be many games giving the same graph, but the graph of a game is unique. We will explore winning strategies in a game by analyzing the corresponding graph.

All the readers have probably tried to play a finite game and they have noticed that there are "winning positions" — when players realize that they are able to win the game, no matter what their opponent moves. Our method will be exactly the same: characterizing the winning positions in the graph.

We are going to associate a non-negative integer to each vertex in the graph such that one can easily create the winning strategies by using these numbers. We write index zero to the ending positions — the vertices which have no successors. The player moving to one of these positions definitely wins. Now, there are some vertices whose successors are already numbered. We will try to index these vertices based on the indices of their successors.

We need to define the *least common excluded* (abbreviated as LCE) of natural numbers:

$$LCE(a_1, a_2, \dots, a_k) \qquad\qquad (a_i \in \{0, 1, 2, \dots\})$$

is the least non-negative integer which is not included in the set $\{a_1, a_2, \dots, a_k\}$.

If the numbers of the successors are $a_1, a_2, \dots, a_k$, then we write the number $LCE(a_1, a_2, \dots, a_k)$ to that vertex.

Theorem 1. *If some vertices have not been indexed yet, then one can find a vertex which can be indexed — i.e. a vertex with all of its successors already indexed.*

Proof. Suppose that none of the vertices can be indexed. Then take a vertex V_1 which is not indexed. It cannot be indexed, therefore V_1 cannot be an ending position. But then there has to be a successor V_2 of V_1 which has not been indexed yet, and so on. Now we have a sequence of vertices: $V_1, V_2, \dots$, where none of the vertices has been indexed yet, and V_{n+1} is the successor of V_n. The game is finite, so there must be a vertex which occurs more than once in the sequence. This means that there is a loop, contradiction.

Due to Theorem 1, a graph of a finite game can be indexed. These indices are called *Grundy-numbers*.

Now we are able to describe how to find all the winning strategies. Suppose that we have already indexed the positions by the procedure above. If anyone is at a vertex with non-zero index, then (s)he can move to a successor of zero index. If anyone is at a vertex of zero index, (s)he can only move to a vertex of non-zero index. Since the ending positions have zero indices, the winning positions are exactly the non-zero positions. The winning strategy is to move always to a successor with zero index. If the starting position is non-zero, than the starting player — White — has a winning strategy; otherwise Black can find a winning way.

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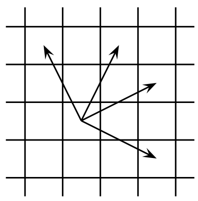
Now we present some examples of finite games, where Grundy-numbers can be used effectively.

The Knight Game

This is a board game, played with one figure moving similarly to the Knight in chess. The starting position is when the Knight is on the bottom left corner of a board with $n \times k$ fields. The following moves are allowed:



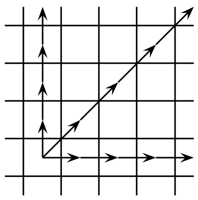
1	**0**	**0**	1	1	**0**	**0**
1	**0**	**0**	1	2	**0**	**0**
2	2	3	2	2	2	1
2	3	4	1	2	1	1
1	**0**	**0**	4	3	**0**	**0**
2	**0**	**0**	3	2	**0**	**0**
2	2	1	2	2	1	1

The game is a finite game, because every step increases the sum of the coordinates, but the sum cannot exceed $n+k$. We can write the Grundy-numbers directly onto the board.

The zero indices have a simple distribution: they are in the 2×2 squares — as in the figure above with thickened numbers. There are no indices bigger than 4, because a position does not have more than four successors, and the LCE of four numbers cannot be bigger than 4.

The Queen Game

The reader is right: this is also a board game, with a figure starting in the same position as before but moving similarly to the Queen in chess. The players may move the Queen only to the right, up or diagonally in these directions:



6	5	4	3	2	1	**0**
7	3	5	4	**0**	2	1
8	4	3	5	1	**0**	2
1	**0**	2	6	5	4	3
9	6	7	2	3	5	4
10	8	6	**0**	4	3	5
3	10	9	1	8	7	6

If you look at the feasible moves, you may notice that the Queen gets nearer and nearer to the top right corner, and at finally it cannot move any further. It means that this is also a finite game. The Grundy-numbers are written on the board.

As a consequence of the structure of the steps, each row, column and diagonal contains exactly one zero index. It is an interesting observation that the zero indices are on two lines with slopes $\dfrac{\sqrt{5}+1}{2}$ and $\dfrac{\sqrt{5}-1}{2}$.

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Up to this point we did not care about the exact value of a non-zero index. Now it comes: the use of Grundy-numbers can be extended.

Finite games can be put together in the following way: Two players play several games at the same time, the player who moves has to choose one of the given games, and then make a move in that particular game. If no move can be made in any of the games, then the game is over. The player who cannot move, loses. If we sum up some games this way, then the game so obtained is called the NIM-sum of the games.

Of course we could try to set up the graph of the NIM-sum — it is also a finite game — but this graph is far more complicated than the graphs of the sub-games we summed up. The idea of a more efficient method for Grundy-numbering is that if you know the indices in the sub-games, then you can calculate the index of the NIM-sum of these games. The index of a “compound” position is equal to the so-called NIM-sum of the corresponding indices in the sub-games.

We define the $\oplus$-sum of natural numbers. We write all the numbers in binary form below each other. We get the $\oplus$-sum digit by digit: Each digit of the $\oplus$-sum can be obtained from the corresponding digits of summands. If the number of 1-s is even, then the resulting digit is 0, otherwise 1. We illustrate the process with an example:

$$\begin{array}{rcccccc} & 1 & 0 & 1 & 1 & 0 & 0 \\ \oplus & & 1 & 1 & 0 & 0 & 1 \\ \hline & 1 & 1 & 0 & 1 & 0 & 1 \end{array}$$

The $\oplus$-sum of more than 2 numbers can be calculated in the same way. It is easy to check that $\oplus$ satisfies the following relations:

(1) $x \oplus y = y \oplus x$

(2) $(x \oplus y) \oplus z = x \oplus (y \oplus z)$

(3) $x \oplus (x \oplus y) = (x \oplus x) \oplus y = 0 \oplus y = y$

(4) $x \oplus 0 = x$

Now, we have defined a special $\oplus$ operation on natural numbers. In order to prove that this operation is suitable for obtaining the Grundy-number of a NIM-sum from the indices of the sub-games, we need the following statement.

Theorem 2.

$$a \oplus b = LCE\,(0 \oplus b,\ 1 \oplus b, \ldots, (a-1) \oplus b,\ a \oplus 0,\ a \oplus 1, \ldots, a \oplus (b-1))$$

Proof. To show this, we have to prove two things:

(i) $s = a \oplus b \notin \{0 \oplus b, \ldots, (a-1) \oplus b, a \oplus 0, \ldots, a \oplus (b-1)\} = H(a,b)$,

(ii) $s' < s \ \Rightarrow\ s \in H(a,b)$.

(i) Suppose that $s \in H(a,b)$. This means $s = a \oplus b'$ $(b' < b)$ or $s = a' \oplus b$ $(b' < b)$ Without loss of generality, we can assume that $a \oplus b = s = a \oplus b'$ $(b' < b)$. By the relations (1) - (4) above, we get

$$b = b \oplus 0 = b \oplus (a \oplus a) = (b \oplus a) \oplus a = (b' \oplus a) \oplus a = b' \oplus (a \oplus a) = b' \oplus 0 = b'$$

But $b \neq b'$, which is a contradiction.

(ii) We have to prove that $s' \in H(a,b)$ whenever $s' < s$. How can we decide whether a number is bigger or smaller than another? We search for the first non-identical binary digit, and the number with the digit 1 is the larger.

Take $s' < s$ and let $d = s \oplus s' > 0$. The first non-zero digit in d stands for 2^α. Since $s' = s \oplus d < s$, the digit for 2^α in number s equals 1.

Now the digit for 2^α in $s = a \oplus b$ equals 1. But this means, that exactly either a or b has digit 1 for 2^α. Assume that this holds for a. Then $a' = a \oplus d < a$.

We have $a' < a$, therefore $a' \oplus b \in H(a,b)$, and

$$a' \oplus b = (a \oplus d) \oplus b = (a \oplus b) \oplus d = s \oplus d = s'.$$

Thus, $s' \in H(a,b)$ for every $s' < s$, which completes the proof of the theorem.

Theorem 3. *The Grundy-number of a position in the NIM-sum of finite games equals the $\oplus$-sum of the Grundy-numbers of the positions in the sub-games.*

Proof. It is enough to prove this for the NIM-sum of two games, because the NIM-sum of more than two games can be attained by adding them up one after the other.

We take two finite games G and H, let their NIM-sum be N. The position in these two sub-games are A and B, the successors are $A_1, \dots, A_k$ and $B_1, \dots, B_n$. The indices of these positions are $a, b, a_1, \dots, a_k, b_1, \dots b_n$, respectively.

If both A and B are ending positions, then the statement is obvious. Therefore, we have to discuss only the case when (A, B) is not an ending position.

Since $A_1, \dots, A_k$ are the successors of A and $B_1, \dots, B_n$ are the successors of B, we have

(5) $\quad a = LCE(a_1, a_2, \dots, a_k)$

(6) $\quad b = LCE(b_1, b_2, \dots, b_n)$

The position in N is (A, B), which is not an ending position, and the successors are $(A, B_1), \dots, (A, B_n), (A_1, B), \dots, (A_k, B)$. To prove the theorem, we have to establish

$$a \oplus b = LCE(a \oplus b_1, \dots, a \oplus b_n, a_1 \oplus b, \dots, a_k \oplus b). \tag{7}$$

First we show that $a \oplus b \geqslant LCE(a \oplus b_1, \dots, a \oplus b_n, a_1 \oplus b, \dots, a_k \oplus b)$. Suppose that the opposite holds, i. e. $a \oplus b = p \in P(a, b)$. This would mean that $a \oplus b = a \oplus b_i$ for some i or $a \oplus b = a_j \oplus b$ for some j, but then:

$$b = b \oplus 0 = b \oplus (a \oplus a) = (b \oplus a) \oplus a = (b_i \oplus a) \oplus a = b_i \oplus (a \oplus a) = b_i \oplus 0 = b_i,$$

which clearly contradicts (6). The case $a \oplus b = a_j \oplus b$ leads to a similar contradiction.

Next we prove that the opposite direction of the inequality also holds. By (5), we get:

$$\{0, 1, 2, \dots, a-1\} \subseteq \{a_1, \dots, a_k\}.$$

This yields

$$\{0 \oplus b, 1 \oplus b, \dots, (a-1) \oplus b\} \subseteq \{a_1 \oplus b, \dots, a_k \oplus b\}$$

and similarly:

$$\{0 \oplus a, 1 \oplus a, \dots, (b-1) \oplus a\} \subseteq \{b_1 \oplus a, \dots, b_n \oplus a\}.$$

It means that the right hand side of (7) contains all the numbers on the left hand sides of the above inclusions: $0 \oplus b, 1 \oplus b, \dots, (a-1) \oplus b,\ 0 \oplus a, 1 \oplus a, \dots, (b-1) \oplus a$. Using an evident property of LCE and Theorem 2., we get

$$\begin{gathered} LCE(a \oplus b_1, \dots, a \oplus b_n,\ a_1 \oplus b, \dots, a_k \oplus b) \geqslant \\ \geqslant LCE\,(0 \oplus b,\ 1 \oplus b, \dots, (a-1) \oplus b,\ a \oplus 0,\ a \oplus 1, \dots, a \oplus (b-1)) = a \oplus b, \end{gathered}$$

which completes the proof.

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A computer program has been developed by M. Makai to illustrate the winning strategies. It contains Heap Games where the two players (You and your computer called SuperMind) reduce the number of elements of the heap or heaps alternately and who is not able to move, loses. Heap Games with two heaps are the same as Board Games, where the players can step like a chess figure, or can choose steps from a collection of pre-defined ones. This way, the user (You) can create his/her own game. The steps can be represented by vectors like the steps of the king and knight or by directions like the steps of the queen and tower. The size of the board is determined by the user as well. The user can also remove some fields from the board.

In the option of NIM-sum of games you can play on more than one board at the same time. You and SuperMind can step on one of the boards alternately. The one who is not able to move at all, loses. If you play a one-board-game, you have to listen to the Grundy numbers; in the case of a several-board-game, you have to look at the NIM-sum of the Grundy numbers. Your moving possibilities are always shown on the screen. SuperMind knows the winning strategies, so it always moves to one of the winning fields, and if it was impossible it moves by chance. You might ask SuperMind to make mistakes, so you can make your job easier, since it is hard enough without knowing the exact Grundy numbers. If the game seems to be too difficult for you, you can always give it up; but it is better to play until the end because you might have already found the winning strategy.

After the conference, the program will be available through the INTERNET by an anonymous ftp from @math.klte.hu. We wish you a good time and good playing.

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References

[1] E. R. Berlekamp – J. H. Conway – R. K. Guy, *Winning Ways*, Academic Press, London, 1982.

[2] L. Csirmaz, *Játékok és Grundy-számaik*, Középiskolai Matematikai Lapok, **61** (10) (1980), 193–212.

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Ramsey-type Problems of Colouring the Real Space*

1. Introduction

By a colouring of some set of objects X in k colours we mean a function

$$f: X \to \{1, 2 \dots, k\}.$$

Given a function of colouring, we can define a partition of the set X which is also called a colouring. It is defined in the following way:

$$i, j \text{ are in the same class iff } f(i) = f(j).$$

* This is a shortened version of the original paper.

For a subset $Y \subseteq X$, we say that it is *monochromatic*, if Y is contained by a class of the given partition, i.e. $f|_Y$ is a constant function.

This strictly formal definition of colouring coincides with our intuitive notion.

Ramsey theory is an important branch of combinatorics. It started with some papers of the English mathematician FRANK P. RAMSEY in 1930. Roughly speaking, its basic idea is that in any 'irregular' colouring of some structure (points in a plane, graphs, arbitrary sets) one can find some 'regular' — monochromatic — sub-structure, supposing that the structure is big enough. A classic example of Ramsey-type problems is the following: show that in any colouring of the sides and diagonals of a regular hexagon (i.e. edges of the complete graph K_6), there is a monochromatic triangle. We assume that the solution of this problem is well-known to any reader. Yet, a bit more complicated problem of the same type is still open: one should determine the minimal number $V(k)$ with the property that in any colouring of a regular V-gon in k colours, there is a monochromatic triangle. It is known that $V(2)=6$, $V(3)=17$, but for $k \geqslant 4$, $V(k)$ is not known (the only result is $47 \leqslant V(4) \leqslant 52$). Also, several bounds for $V(k)$ are found, for example $V(k) \leqslant [n!e]$. One should be aware that this is one of the *simplest* problems of Ramsey theory. Therefore, we can conclude that Ramsey theory is very complex, and it gives a lot of possibilities for further research in combinatorics.

2. Colourings of a straight line and the plane

The following problems are elementary in Ramsey theory.

Problem 1. *A straight line is coloured in two colours. Prove that there is a segment whose ends and midpoint are of the same colour.*

Problem 2. *A plane is coloured in two colours. Prove, that there is a triangle similar to a given one, whose vertices are of the same colour.*

Problem 3. *A plane is coloured in k colours. Prove there is a rectangle whose vertices are of the same colour.*

We can also generalize problems 1. and 2. for k colours, too ($k \geqslant 3$). In this section we are going to present the solution of the generalization of problem 2. and then show how this latter implies the solution of the generalization of problem 1 and problem 3.

Let the plane be coloured with k colours in an arbitrary way. For $k \geqslant 3$, define a sequence (a_n^k) of naturals by

$$a_n^k = a_{n-1}^k \left(k^{\frac{a_{n-1}^k (a_{n-1}^k + 1)}{2}} + 1 \right),$$

where $a_1^k = 2$.

Construct a triangular lattice in the plane with lines parallel to the sides of the given triangle. Fix one of the lines of this triangular lattice. We are going to define a special structure of points, called *binary structure*:

(1) Vertices of a unit triangle of the lattice whose bases lie on the fixed line form a binary structure of order 1,

(2) A binary structure of order $n+1$ is a triangular zone of points which contains two disjoint, identically coloured binary structure of order n with bases lying on the fixed line.

We are going to prove the following statement by induction: a triangular zone with a_n^k points on the base is a binary structure of order n.

For $n=1$, the statement is true. Consider next the inductive step. Inside the zone of dimension a_n^k, we can range

$$k^{\frac{a_{n-1}^k(a_{n-1}^k+1)}{2}}+1$$

zones of dimension a_{n-1}^k. By Dirichlet's principle, there are two zones coloured in the same way (because each of them contains $\dfrac{a_{n-1}^k(a_{n-1}^k+1)}{2}$ points). By our inductive hypothesis, these two zones are binary structures of order $n-1$, so it follows that the observed triangular zone is a binary structure of order n.

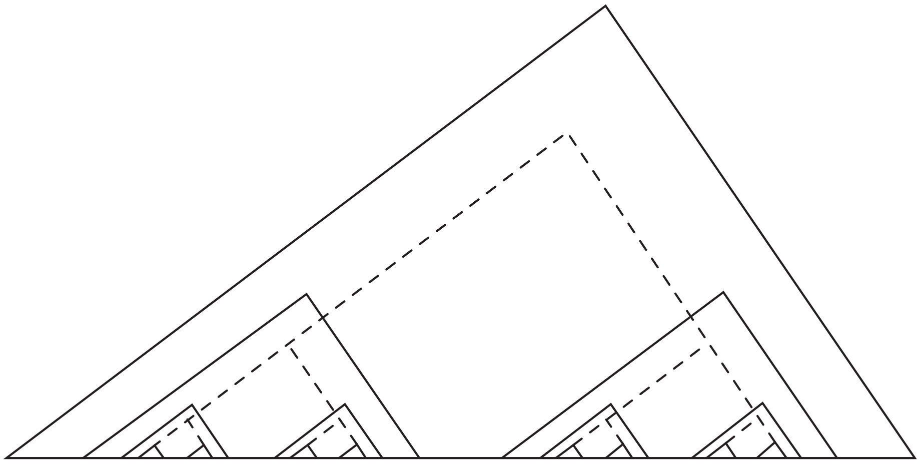

Figure 1

Consider now a triangular zone of dimension a_{k+1}^k. This is a binary structure of order $k+1$. Observe all its substructures of order 1. By the definition of the binary structure, these are all identically coloured. Choose one of the three vertices of the unit structures (see figure 1). Consider a triangular lattice defined with these points and extract a binary tree (see figure 2). This is a binary tree with $k+1$ levels and each of them is monochromatic, by construction. Clearly, there are two levels of the same colour. But then there is a monochromatic triangle similar to the given one, and the proof is finished.

Now, let us consider an arbitrary colouring of the line p and choose the colouring of the plane in the following way: if the point X of the line is coloured in a colour α, then each point X' on the line orthogonal to p in X is coloured in α. We will apply the statement of the generalization Problem 2. Consider such a triangle $\triangle ABC$ that $p(A,B)\,||\,p$ and $CA=CB$. By the previous considerations, one can find a monochromatic triangle in the plane that is similar to the given one. But then the normal projections of

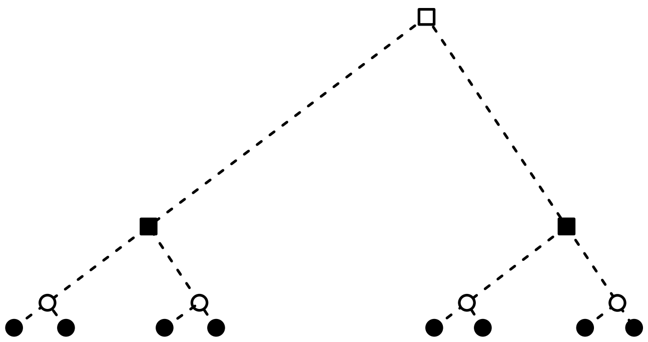

Figure 2

the vertices of that triangle to the line p are of the same colour, by the definition of the special colouring. Therefore they form a structure we called 'arithmetic progression'.

3. Colourings in the space

Naturally, we can pose the question whether it is possible to generalize this problem to the space. Namely, we have the following.

Problem 4. *A tetrahedron and an arbitrary colouring of space is given. Is there a tetrahedron, similar to the given one, whose vertices are of the same colour?*

It turned out that the answer is positive, i.e. the problem can be solved by imitation of the previous proof, with some necessary modifications.

Consider a tetrahedral lattice in the space, whose shape is defined by the given tetrahedron. Fix one plane of this lattice. We define *ternary structures* in an inductive way:

(1) A unit tetrahedron is a ternary structure of order 1.

(2) A ternary structure of order $n+1$ is a tetrahedral zone of points which contains three disjoint, identically coloured ternary structures of order n whose bases lie on its base; where these smaller ternary structures are arranged so that the corresponding vertices form a triangle similar to the base of the given tetrahedron (see figure 3).

We show by induction that there is a finite number $m(k,n)$ such that every tetrahedral zone with $m(k,n)$ points on the base contains a ternary structure of order n.

For $k=1$ the statement is true, because $m(k,1)=3$. Put $m=m(k,n)$.

Let $f:\mathbf{N}\rightarrow\mathbf{N}$ be a function with the next property: if a tetrahedron has l points of the lattice in the base, then the total number of integer points inside the tetrahedron is $f(l)$. Let (a_n^k) be the sequence from the previous section and $q=k^{f(m)}$. Consider tetrahedral zones of dimension m on the plane, already fixed, in the way presented by figure 4, with a_{q+1}^q tetrahedrons along the edges.

We transform each tetrahedron into a single point (for example, into its mass-center), with extending the number of colours to q ("compound" colours). By considerations from

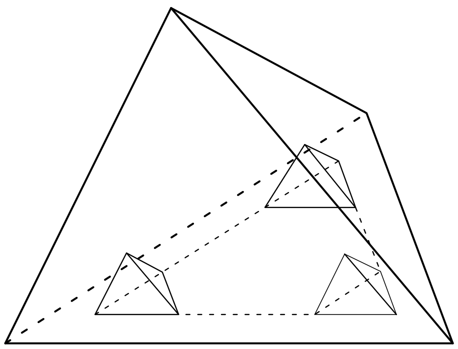

Figure 3

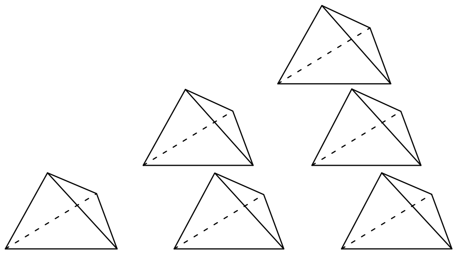

Figure 4

the previous section, we can find a monochromatic triangle, which means that we find three tetrahedral zones that are of the same colour and arranged in the way required in the definition of a ternary structure. Since each of the observed tetrahedrons contains a ternary structure of order n, we have a ternary structure of order $n+1$. Having so, the statement is proved, because

$$m(k, n+1) = m\frac{a_{q+1}^{q}(a_{q+1}^{q}+1)}{2}$$

is an appropriate choice.

Consider a ternary structure of order $k+1$, which exists by the previous statement, and its substructures of order 1. They are all of the same colour by the definition of the ternary structure. Choose one of the four vertices of the unit tetrahedron. Consider a tetrahedral lattice defined with these points and extract a ternary tree (see figure 5).

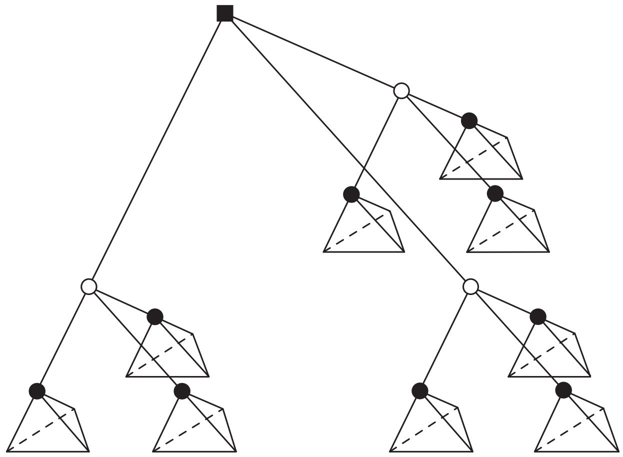

Figure 5

By the construction, this ternary tree has $k+1$ levels and all points on the same level are of the same colour; but then we have two levels equally coloured, so we have a monochromatic tetrahedron similar to the given one.

Finally we show how this solution implies the solution of a similar problem.

Problem 5. *Prove that there is a monochromatic quadrilateral similar to a given one in any colouring of the plane.*

Let α be the plane where we are looking for such a monochromatic quadrilateral. Consider a special colouring of the space: if a point $X \in \alpha$ is of the colour β_i $(1 \leqslant i \leqslant k)$ and $\nu(x)$ is the line orthogonal to α in X, then colour all the points of the line ν with β_i.

Choose a tetrahedron $ABCD$ such that its normal projection on the plane α is congruent to the given quadrilateral. By the solution of problem 4., there is a monochromatic tetrahedron similar to $ABCD$. But then its normal projection on α is the desired monochromatic quadrilateral, by the definition of the special colouring.

Acknowledgement. The problems exposed in this text were presented as a part of the competition program of the final round of the 16^{th} International Tournament of the Towns, that was held in Novi Sad (Yugoslavia) 1–9 August, 1995. There were 20 towns from 6 countries (Yugoslavia, Russia, Ukraine, Great Britain, Bulgaria and Israel) participating in the final round. The team of Novi Sad, in which M. Vuletić took part, received 2^{nd} prize for the solutions of the set of problems "Colourings".

Mirjana Vuletić,
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Igor Dolinka,
Faculty of Science, University of Novi Sad

Problems of Points Competitions of KöMaL

that appeared in the 1995–96 school year

Problems are assembled by the mathematical and physical boards. One can find university professors as well as students as well among the submitters. Solutions appear on the pages of the journal in 1996 and 1997 in Hungarian language.

Mathematics

Exercises for practice:

C. 401. Matthew is in a constant hurry. He noticed that standing on the escalator it took him a minute and a half to get down to the metro, but he could run down on the escalator in one minute when it was out of action. How long would it take him to run down on the moving escalator?

C. 402. Is it possible to choose parameters a, b and c so that equality

$$(x+a)^2+(2x+b)^2+(2x+c)^2=(3x+1)^2$$

holds for every x?

C. 403. Prove that a triangle can be divided by a line segment into two triangles, both similar to the original one, if and only if it is a right triangle.

C. 404. We are given two regular rectangular pyramids, both of them have side faces of unit height. The lengths of their side edges are 1.25 and 1.33, respectively. Which one has the larger volume?

C. 405. Find digits A, B, C and D so that $ABC \cdot AD = ADDC$.

Submitted by E. Fried, Budapest

C. 406. There are given 100 positive integers. Show that either one of them is divisible by 100 or some of them add up to a number which is divisible by 100.

Submitted by Gy. Kósa, Kiskőrös

C. 407. The outer angle bisectors of a triangle ABC form an other triangle the angles of which measure 40°, 65° and 75°. Find the angles of the original triangle.

Submitted by Zs. Merétey-Vida, Budapest

C. 408. We build a cube using 27 unit cube pieces. Is it possible to remove 10 unit cubes from the building so that the surface area of the remaining body is the same as that of the large cube? (We assume that the relative position of the cubes we have after the removal remains the same.)

C. 409. In 1994, the deficit of the foreign trade balance was 3.8 billion dollars. According to the report of the Economy Research Inc., the deficit would decrease to 3 billion dollars in 1995, assuming an 11% increase of the export and a 3% increase of the import. Calculate the amount of the export and the import in 1994, respectively.

C. 410. Adding up the natural numbers from 1 to n, the sum, in certain cases, is divisible by a (positive integer) power of 10. Find the smallest n for which the sum is divisible by 10000.

C. 411. Let n be a positive integer, and let $\alpha = 180/(1+2^n)$. Prove equality

$$(\cos\alpha)\cdot(\cos 2\alpha)\cdot(\cos 4\alpha)\cdot\ldots\cdot\left(\cos 2^{n-1}\alpha\right)=\frac{1}{2^n}.$$

Submitted by R. Láng, Balatonfüred

C. 412. We build a cube using 27 unit cube pieces. We remove some unit cubes from the building, without affecting the relative position of the remaining ones. Find the maximum possible surface area a body arising this way may have.

C. 413. Waiting at the railway crossing, we were observing a locomotive approaching at a speed of 20 m/s. The engine reached the crossing 4 seconds after we began to hear it hooting. How far was the engine from the crossing when it started hooting? (The speed of the sound is 340 m/s.)

C. 414. Let us define the strange numbers on the following way. Every one-digit prime number is strange, and an at least two-digit prime number is strange if and only if so are the two numbers obtained from it by omitting either its first or its last digit. Find all strange numbers.

Submitted by E. Fried, Budapest

C. 415. The vertices of a kite are $A(0;0)$, $B(1;2)$, $C(4;0)$ and $D(1;-2)$. Calculate the radii of those circles the circumference of which lies in the same distance from all four vertices.

C. 416. We are given a triangle ABC. Consider those tetrahedra $ABCP$ the shortest altitude of which is the one drawn from vertex P. Determine the locus of points that are obtained from such points P by an orthogonal projection to plane ABC.

C. 417. A four-digit number has the following properties:
I. The sum of the first and the second digits (i.e. those that represent the number of thousands and hundreds, respectively) is equal to the sum of the last two digits.
II. The sum of the second and the fourth digits is twice the sum of the first and the third digits.
III. Adding up the first and the fourth digits we receive the third one.
IV. The sum of the second and the third digits diminished by the first one equals three times the last digit. Do these properties determine the number exclusively?

C. 418. Which perfect squares have a binary representation that consists of digits 1 only?

C. 419. The lengths of the legs of a right triangle are 126 and 168 units, respectively. Find the perimeter of the triangle formed by the feet of the angle bisectors of the original triangle.

C. 420. We are given a regular hexagonal pyramid. The angles formed by its side faces and its base measure 45°. Lay a plane S incident to one of the base edges so that it intersects the side faces corresponding to the base edges that are neighbours to the chosen edge at two parallel segments. Find the measure of the angle formed by the plane S and the base of the pyramid.

C. 421. A certain wall-clock cannot be wound up when any of its two hands is between 3 and 4 or is between 8 and 9. What is the daily accumulation of the periods when we can wind the clock up?

C. 422. Find the positive integer solution of the inequality $n^3 - n < n!$. ($n!$ means the product of the first n positive integers.)

C. 423. In a right triangle, the touching point of the incircle divides the hypotenuse into parts of length x and y, respectively. Prove that the area of the triangle is xy.

C. 424. We cut a circular piece of paper into n congruent sectors, and create a cone from each of them. Find the value of n for which the sum of the volumes of the cones is maximum.

C. 425. There is a 3 digit number, written in decimal system. Its reverse is a number made up of the same digits in reverse order. Dividing the number by its reverse, the partial quotient is 3, and the remainder equals the sum of the digits of the number at issue. Find the number.

Submitted by I. Varga, Békéscsaba

C. 426. The first term of a geometric progression is 2^n, where n is a positive integer, and its common ratio is $3/2$. Prove that the sum of the first $n+1$ terms of the sequence is divisible by 5 if and only if n is odd.

C. 427. A given circumscribed trapezium has a right angle. Draw a line parallel to its bases through the intersection point of the diagonals. Show that the length of its segment lying between the legs of the trapezium is equal to the height of the trapezium.

Submitted by L. Németh, Fonyód

C. 428. There are 4 pairwise disjoint spherical calottes placed on a sphere. At most what portion of the surface area of the sphere is covered by the calottes?

C. 429. Consider the sequence defined by

$$a_k=\frac{a_1}{1+(k-1)a_1} \qquad (a_1>0;\ k=1,\,2,\,\ldots,\,n).$$

Derive equality

$$a_1a_2+a_2a_3+\ldots+a_{n-1}a_n=(n-1)a_1a_n.$$

Submitted by G. Bakonyi, Budapest

C. 430. Let t denote a fixed real number. Solve equation

$$x^3-2tx^2+t^3=0.$$

C. 431. We are given a triangle OXY and a positive number λ. Choose a point Q on segment XY, and emerge a line perpendicular to OQ at point O. Consider those points P on this line for which $OP/OQ=\lambda$. What is the locus of P if Q ranges over XY?

Submitted by I. Ratkó, Budapest

C. 432. Rotate a square about one of its axes of symmetry. Let us denote by F resp. V the surface area and the volume of the body obtained this way. Find all possible values of the ratio F^3/V^2.

C. 433. Let n denote a positive integer. Prove that it cannot happen that both $n+1$ and $4n+1$ are perfect squares.

Submitted by I. Varga, Békéscsaba

C. 434. Solve the inequality $\log_a(x-a)>\log_{1/a}(x+a)$.

C. 435. In the middle of a park, shaped like a square of side length a, there is a circular flower bed of diameter d. There are paths along the edges of the park as well as around the flower bed. There are also straight paths connecting the vertices of the park and the midpoints of the edges of the park to the flower bed towards the center of the circle, respectively. Find the shortest walk that connects one corner of the park to the opposite corner.

Submitted by E. Fried, Budapest

C. 436. We are given the frustum of a regular rectangular pyramid. The altitude of the frustum is the same as the common length of the edges of the top face. The volume of the frustum is twice as large as the volume of the largest cube that can be inscribed into it. Find the measure of the angle formed by a side face and the base of the pyramid.

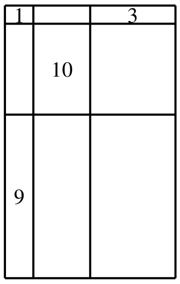
Exercises (for 14–16 year-old students):

Gy. 2998. The sequence $a_0,\,a_1,\,\ldots$ is defined by recursion

$$a_0=1995, \qquad a_n=na_{n-1}+1, \quad \text{ha } n\geqslant 1.$$

Upon dividing a_{2000} by 9, find the remainder.

Submitted by J. Pataki, Duino



Gy. 2999. A rectangle is divided into 9 smaller ones, the areas of four of which are indicated on the Figure. Prove that the area of the rectangle is at least 90 unit.

Gy. 3000. Find the real solutions of the system of equations

$$|y-x|-\frac{|x|}{x}+1=0;$$
$$|2x-y|+|x+y-1|+|x-y|+y-1=0.$$

Submitted by S. Mihalovics, Esztergom

Gy. 3001. Is it possible to place the 16 white and 16 black pieces on the chessboard so that each piece has the same number of black as white neighbours?

Submitted by Z. Blázsik, Szeged

Gy. 3002. The sides of a triangle have lengths a, b and c. The real numbers x, y and z add up to 0. Derive inequality $a^2yz+b^2zx+c^2xy\leqslant 0$.

Gy. 3003. We are given two discs of different radii. Construct a right triangle with the following property: considering the legs of the triangle as the diameters of two new discs, the sum and the difference of the areas of these new discs is equal to the area of the bigger and the smaller disc, respectively. *Submitted by G. Bakonyi,* Budapest

Gy. 3004. Each vertex of a regular 7-gon is coloured either red, blue, green or yellow, according to the following rules. If a vertex is coloured either red or blue, then the colour of its first and fourth successor (in clockwise orientation) is neither blue nor green, and if a vertex is coloured either yellow or green, then the colour its first and fourth successor is neither red nor yellow. Prove that every vertex has the same colour.

Submitted by Dragan Marusic, Ljubljana

Gy. 3005. We are given a right cylinder, both its altitude and its diameter are of unit length. Starting at its base, we intersect the cylinder with planes parallel to the base so that the distance between two consecutive planes is always 0.01 unit. Inscribe a regular hexagon into each of the 101 circles arising this way so that neighbouring hexagons are obtained from each other using a rotation of 30° about the axis of the cylinder. The vertices of the hexagons form the vertex set of a polytope, the base faces of which are two hexagons, and the side faces of which are such triangles that are formed by two consecutive vertices of some hexagon and the vertex of a neighbouring hexagon that lies between the first two vertices. Show that the surface area of this polytope is greater than that of the cylinder.

Gy. 3006. There are a few pieces placed on an $n\times k$ chessboard (n, $k\geqslant 2$). (Each piece is placed on a certain field of the board, no two of them standing on the same field.) Choosing any two rows and 2 columns there is always an odd number of pieces standing on the 4 fields formed by their intersection. Find the best lower and upper bounds for the number of pieces that can be placed this way. *Submitted by Z. Blázsik,* Szeged

Gy. 3007. We define a sequence a_n by the following recursion: $a_1=1$, $a_2=2$ and $a_n=a_{n-1}+a_{n-2}$ for $n\geqslant 3$. Prove inequality

$$\left|\frac{a_{n+1}}{a_n}-\frac{a_{k+1}}{a_k}\right|\leqslant 1$$

for integers n, $k\geqslant 1$. *Submitted by T. Kerekes,* Budapest

Gy. 3008. We arrange integers in an infinite tabular form as follows. The first column is an arithmetic progression the first term of which is 4 and the common difference of which is 3. For every $k \geqslant 1$, the k-th row is an arithmetic progression of common difference $2k+1$. Where can one find the numbers 1994 and 1995 in this arrangement?

Submitted by J. Pataki, Duino

Gy. 3009. There are 1995 discs lying in a row. One side of each disc is red, the other one is blue. We may repeatedly move the discs. At one move we are allowed to turn over either m or n neighbouring discs. Is it possible to obtain a row of red discs from any starting position if
a) $m=19$ and $n=94$ or b) $m=19$ and $n=95$?

Gy. 3010. Let a, b, c denote the sides of a triangle. Does equality $a^3+b^3=c^3$ imply that the triangle is an acute one?

Gy. 3011. Construct a trapezium if there are given the lengths of its bases and one of its legs, and the ratio of its diagonals.

Gy. 3012. Two concentric circles have radii of 1 and 2, respectively. Find the minimum of an area a quadrilateral may have if it is inscribed into the larger circle and contains the smaller one.

Gy. 3013. A tourist intends to climb a frustum-of-a-cone shaped volcano. The height of the volcano is 1000 m, the radii of its bases are 1500 m and 750 m, respectively. Find the minimum distance the tourist should cover from the foot of the hill to the top if he wants to get around the hill at the same time.

Gy. 3014. Prove that if for numbers a, b, c, d $a>b>c>d$ holds then

$$c<\frac{cd-ab}{c-a+d-b}<b.$$

Gy. 3015. The black king is placed at the left upper corner of an $n\times m$ chessboard. Two players move alternately. Each player may move the king to a neighbouring field that has not yet been visited by the king. The one who cannot move, loses. Which player has a winning strategy?

Gy. 3016. The real numbers a_0, a_1, ..., a_n satisfy the following conditions: $a_0=a_n=0$ and $a_{k-1}+a_{k+1}-2a_k\geqslant 0$ $(k=0, 1, \dots, n-1)$.
Prove that $a_k\leqslant 0$ for every k.

Gy. 3017. Find all solutions (p, q, n) of equation

$$n^2=p^2+q^2+p^2q^2$$

where p and q are prime numbers and n is a natural number.

Submitted by I. Varga, Békéscsaba

Gy. 3018. The relative length of a side of a convex polygon is defined by the ratio of the length of the side and the length of the longest chord of the polygon, parallel to the given side. Show that every convex quadrilateral has a side of relative length at least 1, and a side of relative length at most 1, too.

Gy. 3019. The quadrilateral $ABCD$ can be inscribed in a circle. Prove equality

$$AC\cdot(AB\cdot BC+CD\cdot DA)=BD\cdot(BA\cdot AD+BC\cdot CD).$$

Submitted by S. Róka, Nyíregyháza

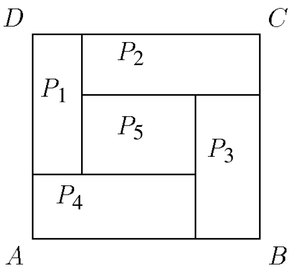
Gy. 3020. Given an acute triangle ABC, construct the equilateral triangle of maximum area the sides of which are incident to points A, B and C, respectively.

Gy. 3021. Two balls of unit radius intersect each other in such a fashion that the three parts arising by their intersection have equal volumes. Find the distance between the centers of the balls.

Gy. 3022. There is a single-storey hotel at the seaside, with all of its 20 rooms looking out on the bay. A guest at the hotel may occupy either two adjoining rooms for one day, or one single room for two consecutive days. A room costs 1 golden gulden per day. According to the visitor's book, Room 1 was empty on the first day of the season. On the 100th day, which was the last day of the season, nobody took out Room 20. Prove that the takings of the hotel in that season were at most 1996 golden guldens.

Gy. 3023. Find the integer solutions of the equation $\sqrt{a}+\sqrt{b}=\sqrt{c}$.

Submitted by E. Fried, Budapest



Gy. 3024. A rectangle $ABCD$ is partitioned into rectangles P_1, P_2, P_3, P_4, P_5, according to the Figure. We know that rectangles P_1, P_2, P_3, P_4 have equal areas, and that P_5 is a square. Prove that rectangle $ABCD$ is a square, too.

Gy. 3025. Find all functions $f:R\to R$ that satisfy equation $f\left((x-y)^2\right)=(f(x))^2-2xf(y)+y^2$ for all real numbers x and y.

Gy. 3026. Is it possible to select 1995 non-collinear points in the plane so that any straight line joining two of them passes through at least another one of the selected points?

Gy. 3027. We are given an equilateral triangle $A_1B_1C_1$ with positive orientation. Rotating it about its centroid by $+23^\circ$, and enlarging the image in ratio $\sqrt{5}/2$ from the same point, we obtain an other equilateral triangle $A_2B_2C_2$. Show that there exists a point P so that both $\triangle PA_1A_2$ and $\triangle PB_2C_1$ are equilateral triangles.

Gy. 3028. Given a square of side length 3, draw a circle of radius 3 around each of its vertex. Calculate the area of the intersection of the four circles.

Submitted by G. Nyitray, Győr

Gy. 3029. We are given a regular icosahedron, having three red vertices. Show that it has a vertex that has at least two red neighbours.

Gy. 3030. Find all perfect squares which, in decimal representation, have 444 as their last three digits. Does any of those numbers terminate in 4444?

Gy. 3031. Prove inequality

$$\frac{1}{\sqrt[k]{n+1}}+\frac{1}{\sqrt[n]{k+1}}>1$$

for arbitrary integers k, $n>1$.

Gy. 3032. There are ten money-boxes on a table, each containing a few coins. We may change the position of the coins using any one of the following two methods. One way is to remove one coin from each of some nine boxes and put these 9 coins into the tenth container. The other possibility is just its opposite: remove 9 coins from one of the boxes and distribute them among the remaining 9 boxes, putting one coin into each. It is allowed to have an empty box after performing a step, but, as a matter of fact, one cannot remove any coin from such a container. Initially, there are 1, 2, 3, 4, 4, 5, 5, 11, 12 and 13 coins in the boxes, respectively. Is it possible to arrange that any two boxes contain a different number of coins?

Gy. 3033. Each multiplier denoted by a star in the expression

$$(*)\cdot 8^7+(*)\cdot 8^6+(*)\cdot 8^5+(*)\cdot 8^4+(*)\cdot 8^3+(*)\cdot 8^2+(*)\cdot 8+(*)\cdot 1$$

may be either $+1$ or -1. Two players move alternately, giving a value to one multiplier at a time. Prove that no matter how the first player moves, at the end the second player can obtain a number that is divisible by 13.

Gy. 3034. There is a broken line-segment lying inside a rectangle so that any straight line parallel to a side of the rectangle has at most one point in common with it. Show that the length of the broken line-segment is less than the sum of the lengths of two neighbouring sides of the rectangle.

Gy. 3035. The incircle of triangle ABC touches the sides of the triangle at points P, Q and R, respectively. Prove that the area of triangle PQR is not greater than a quarter of the area of triangle ABC.

Submitted by P. Hanyecz, Budapest

Gy. 3036. Given a circumscribed quadrilateral $ABCD$, let M denote the point of intersection of sides AD and BC. Suppose that distances MA, MB, MC and MD, in this order, form a geometric progression. Given that $\angle BAD=90°$, conclude that the common ratio of this progression is less than 2.

Submitted by G. Bakonyi, Budapest

Gy. 3037. We are given a circle k and a point P outside the circle, in the same plane. If a sphere is incident to k, then the points of contact of the tangents drawn from P to the sphere form a circle. Find the locus of the centers of these circles.

Gy. 3038. Given 50 different positive integers, none of them is greater than 100. Prove that some (maybe only one) of them have a sum that is a perfect square.

Gy. 3039. Find the positive integer solutions of equation

$$\left(1+\frac{1}{a}\right)\left(1+\frac{1}{b}\right)\left(1+\frac{1}{c}\right)=2.$$

Gy. 3040. Let a and b denote arbitrary real numbers. Derive inequality

$$\frac{a^4+a^2b^2+b^4}{3}\geqslant\frac{a^3b+b^3a}{2}$$

Gy. 3041. Find all real numbers x that satisfy equation

$$\frac{1}{1998}\left(\sqrt{2\sqrt{2}x-x^2-1}+\ldots+\sqrt{2\sqrt{2}x-x^2+k^2-2}+\ldots+\sqrt{2\sqrt{2}x-x^2+1995^2-2}\right)=1995.$$

Gy. 3042. Let n be a positive integer. Given a rectangular coordinate system, find the number of different rectangles the vertices of which have nonnegative integer coordinates less than n. (This sides of the rectangles must be parallel to the axes.) Determine also the sum of the perimeters of these rectangles.

Gy. 3043. A circumscribed trapezium $ABCD$ has bases AB and CD ($AB > CD$). Its inscribed circle touches sides AB and CD at points K and L, respectively. Prove that the line connecting the intersection point of AL and KC with the intersection point of legs AD and BC is parallel to the bases of the trapezium.

Submitted by L. Németh, Fonyód

Gy. 3044. A collection of subsets of a finite set H has the following two properties:
a) every pair of two distinct elements of H is contained in a unique distinguished subset of H;
b) for any distinguished subset R and an element p in its complement there are exactly m (n) distinguished subsets that contain p and intersect R (do not intersect R).
Express, in terms of m and n, the cardinality of H and the number of its distinguished subsets.

Gy. 3045. In a tetrahedron, each plane tangent to the inscribed sphere and parallel to a face of the tetrahedron cuts a small tetrahedron off the original one. May some of these small tetrahedra share a common inner point?

Gy. 3046. Is it possible to partition 10000 pebbles into 100 heaps so that each heap contains a different number of pebbles, but if any one of the heaps is further divided into two parts, there always can be found two heaps of equal size among the 101 heaps obtained this way?

Gy. 3047. Find the integer solutions of the equation

$$x^3 + y^3 = 8^{30}.$$

Gy. 3048. a) Enumerate the ways $2n$ people can be partitioned in pairs. (Two pairings are different if there is a person who has different pairs in the two pairings.)
b) Prove that $((nk)!)^2$ is divisible by $(k!)^{n+1}(n!)^{k+1}$.

Gy. 3049. The functions $f_1(x) = a_1x^2 + b_1x + c_1$ and $f_2(x) = a_2x^2 + b_2x + c_2$ have disjoint graphs. The product of their leading coefficients is negative, i.e, $a_1a_2 < 0$. Show that the graphs can be separated by a straight line.

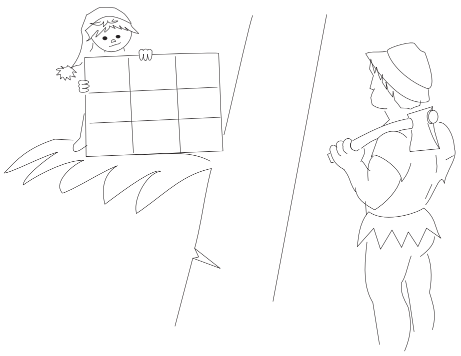
Gy. 3050. One side of a triangle is divided into three parts by the trisectors of the opposite angle. May it happen that the longest one of the three segments is the middle one?

Gy. 3051. A closed polygon, or broken line segment, has the following properties. It cuts each of its segments exactly once, and no three segments of its share a common point. Can such a polygon be made up of
a) 1996, b) 1997 line segments?

Gy. 3052. Given a circle k, let P denote the intersection point of chords AB and CD. We know that P lies inside the circle but differs from its center. Let E and F denote the intersection points of the lines tangent to k at points A and B resp. C and D. Show that any line connecting the touching points of the tangents drawn to k from a point of line EF passes through P.

Gy. 3053. A cuboid is cut into 2, 4, then 8 cuboid pieces by three successive cuts. The total surface area of the cuboids increases at the same ratio during each cut. Prove that the volume of the original cuboid is twice the volume of the largest cube contained in it.

Submitted by G. Bakonyi, Budapest



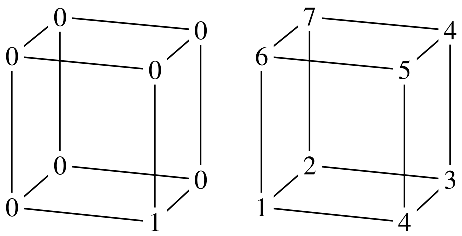
Gy. 3054. A logger was about to drive his axe into the next pine-tree, when suddenly an elf appeared in front of him, crying: "Wait! If you would like to cut this tree, you must solve my puzzle first. Here is this wooden plate, divided into three rows and three columns. You have to fill its fields with the letters S A W I N G S E T such a way that consecutive letters are placed in neighbouring fields, i.e. fields that share an edge. And be careful: No row or column may contain more than one vowel." Help the logger to solve the puzzle, or prove that it cannot be solved.

Gy. 3055. The integers a and b satisfy equality

$$a = a^2 + b^2 - 8b - 2ab + 16.$$

Prove that a is a perfect square.

Gy. 3056. We assigned a number to each vertex of a cube, according to the Figure. We are allowed to choose an arbitrary edge of the cube and increase the number at each vertex of the edge by 1. Applying this step repeatedly, is it possible to achieve that all the numbers assigned to the vertices of the cube are the same?



a) b)

Gy. 3057. Let n denote a positive integer, and let a and b be positive real numbers. Derive inequality

$$\frac{1}{a+b} + \frac{1}{a+2b} + \frac{1}{a+3b} + \ldots + \frac{1}{a+nb} < \frac{n}{\sqrt{a(a+nb)}}$$

Submitted by F. Bodon, Budapest

Gy. 3058. Let α, β, γ, and δ denote the angles of a quadrilateral, in this order. Prove that the quadrilateral is an inscribed one if and only if relation

$$\alpha\beta + \alpha\delta + \gamma\beta + \gamma\delta = \pi^2$$

holds.

Submitted by Z. Nógrádi, Budakalász

Gy. 3059. In an acute triangle ABC, $\angle BAC = 60°$. A circle constructed on side BC with BC as its diameter intersects sides AB and AC at points D and E, respectively. Find the ratio of the area of quadrilateral $BDEC$ and the area of triangle ABC.

Submitted by I. Varga, Békéscsaba

Gy. 3060. Construct a triangle if there are given its inradius, its circumradius, and the distance between its orthocentre and one of its vertices.

Submitted by H. Lancsa, Szeged

Gy. 3061. The boundary of a convex polyhedron consists of 12 equilateral triangles of unit sides and two regular hexagons. The hexagonal faces are parallel to each other. Prove that the polyhedron has a circumscribed sphere, and find the radius of that sphere.

Gy. 3062. At a certain excursion there were n participating couples. Some of them shook hands with others (but not with their own spouses) upon arrival. When the whole team was together, the guide of the excursion asked from each participant the number of people that participant shook hands with. Surprisingly, each of them gave a different answer. Find out whether the guide or his wife shook hands with the more people.

Submitted by D. Papp, Budapest

Gy. 3063. Let p and q be prime numbers. Find the number of pairs of positive integers x, y that satisfy equation

$$\frac{p}{x} + \frac{q}{y} = 1.$$

Submitted by I. Varga, Békéscsaba

Gy. 3064. Let x and y denote arbitrary real numbers. Find the smallest possible value of expression

$$4 + x^2y^4 + x^4y^2 - 3x^2y^2.$$

Gy. 3065. Prove that there is no real function f that satisfies relations

$$f(1 + f(x)) = 1 - x, \qquad f(f(x)) = x$$

for every real number x.

Gy. 3066. Given that a, b and c denote the sides of a triangle, prove inequality

$$(a + b + c)^2 < 4(ab + bc + ca).$$

May we replace the multiplicative factor 4 by 3.9 on the right hand side of the inequality?

Gy. 3067. There are given three parallelograms $A_1A_2A_3A_4$, $B_1B_2B_3B_4$ and $C_1C_2C_3C_4$ in the plane. Let us denote by S_i the centroid of triangle $A_iB_iC_i$. Prove that quadrilateral $S_1S_2S_3S_4$ is also a parallelogram.

Gy. 3068. In a triangle, we are given a point on each side of the triangle, and we are also given three concurrent rays e, f, g that are incident to the three vertices of the triangle, respectively. Construct the triangle.

Gy. 3069. There are given 1996 points in the (3 dimensional) space, no four of them are coplanar. Any four of the given points determine a tetrahedron of volume less than 0.037. Prove that there exists a tetrahedron of volume 1 that contains all the given points in its interior.

Submitted by L. Czirok, Szekszárd

Problems (for 17–18 year-old students)

F. 3076. Find the integer solutions of the system of equations

$$\begin{aligned} x_1^2 &= 1 + x_{1994}x_{1995}; \\ x_2^2 &= 1 + x_{1995}x_1; \\ x_3^2 &= 1 + x_1x_2; \\ &\vdots \\ x_{1995}^2 &= 1 + x_{1993}x_{1994}. \end{aligned}$$

F. 3077. Find all polynomials P satisfying $P(x+1)=P(x)+2x+1$.

F. 3078. We cast a die 10 times. Find the probability of obtaining the sequence 1, 2, 3 of three consecutive casts.

F. 3079. Line segments AB and CD are two perpendicular chords of a circle, neither of them is a diameter of the circle. CD divides chord AB in the ratio $1:5$ and the longer arc AB in the ratio $1:2$. Find the ratio in which chord CD is divided by AB.

F. 3080. A triangle has an inscribed circle of radius $3\sqrt{2}$ that passes through the points of trisection of a median. Find the sides of the triangle.

Submitted by S. Mihalovics, Esztergom

F. 3081. Both sides a and b of a rectangle are integers. Prove that the cuboid of edges a, b, c has an integer diagonal for some positive integer c if and only if the rectangle has an even area.

Submitted by S. Mihalovics, Esztergom

F. 3082. Find those real numbers x for which $[5x]=[3x]+2[x]+1$ holds. ($[x]$ denotes the integer part of x.)

F. 3083. Find the real solutions of equation

$$\frac{1}{x^2}+\frac{1}{x^6}+\frac{1}{x^8}+\frac{1}{x^{12}}-\frac{4}{x^7}=0.$$

Submitted by P. Hanyecz, Budapest

F. 3084. Prove that every integer power of 2 has a multiple the decimal expression of which contains digits 1 and 2 only.

Submitted by I. Farkas, Budapest

F. 3085. The convex polygon $A_1A_2\ldots A_n$ is inscribed into a circle. Let A be a point of its circumcircle, different from the vertices. Set $a_i=AA_i$ ($i=1, 2, \ldots, n$) and let b_i denote the distance of A from line A_iA_{i+1} ($A_{n+1}=A_1$). Derive inequality

$$\frac{a_1^2}{b_1}+\frac{a_2^2}{b_2}+\ldots+\frac{a_n^2}{b_n}\geqslant 2nR,$$

where R stands for the circumradius.

F. 3086. Given a triangle ABC, let P and Q denote the points of the closed triangle for which expressions $PA+PB+PC$ resp. $QA^2+QB^2+QC^2$ admit their maximum values. Show that $P=Q$.

Submitted by J. Pataki, Duino

F. 3087. Consider a cuboid of unit diagonal. Prove that the measures of all of its edges, the double of its surface area and the quadruple of its volume add up to a number smaller than 12.

Submitted by Z. Blázsik, Szeged

F. 3088. The initial term of a sequence is 1. The other terms of the sequence are obtained by adding 4 to the quadraple of the square root of the sum of the preceeding terms. Find the sum of the first 1995 terms of the sequence.

"Hajós György" Mathematical Competition, 1995.

F. 3089. Does the equation $x^2+3xy-2y^2=122$ admit any integer solution?

Submitted by S. Róka, Nyíregyháza

F. 3090. The entries of a 5×5 matrix are real numbers. Any 3×3 (connected) submatrix has the property that the sum of its entries is positive. Show that there can be found a 2×2 (connected) submatrix having the same property.

Submitted by S. Róka, Nyíregyháza

F. 3091. In an acute triangle of area t, let s_a, s_b and s_c denote the medians drawn to sides a, b and c, respectively. Prove inequality

$$\frac{1}{s_a^2-\frac{a^2}{4}}+\frac{1}{s_b^2-\frac{b^2}{4}}+\frac{1}{s_c^2-\frac{c^2}{4}}\geqslant\frac{3\sqrt{3}}{2t}.$$

Submitted by T. Kerekes, Budapest

F. 3092. Five circles, each of them are of radius r, are drawn in the plane. One of them is touched by the other four circles, respectively. The interiors of the circles are pairwise disjoint. Find the radius of the smallest circle that contains all the five circles.

Submitted by P. Hanyecz, Budapest

F. 3093. Suppose that a plane transformation maps the vertex set of any rectangle to the vertex set of a rectangle. Conclude that the transformation is a similarity.

Submitted by L. Pap, Szeged

F. 3094. Find the integer solutions of the equation $\sqrt[3]{a}+\sqrt[3]{b}=\sqrt[3]{c}$.

Submitted by E. Fried, Budapest

F. 3095. The product of the positive numbers a, b, c is 1. Derive inequality

$$\frac{1}{a^2(b+c)}+\frac{1}{b^2(c+a)}+\frac{1}{c^2(a+b)}\geqslant\frac{3}{2}.$$

F. 3096. We are given a polynomial of degree 10, having integer coefficients. We know that 1 is a root of the polynomial of multiplicity at least 8. Prove that the polynomial has a coefficient the absolute value of which is greater than 27.

F. 3097. A convex quadrilateral $ABCD$ is inscribed in a unit circle. Its sides satisfy inequality

$$AB\cdot BC\cdot CD\cdot DA\geqslant 4.$$

Show that the quadrilateral is a square.

Slovakian problem

F. 3098. Let us denote by α, β, γ the angles of a triangle, by $2s$ its perimeter, and let R and r denote its circumradius and its inradius, respectively. Prove relation

$$4R^2\cos\alpha\cos\beta\cos\gamma=s^2-(r+2R)^2.$$

Submitted by G. Reményi, Budapest

F. 3099. A plane cuts each edge of a (non-planar) quadrilateral $ABCD$ at an inner point. According to a fixed orientation, we assign to every side of the quadrilateral the ratio in which the plane divides the given side. Calculate the product of the four ratios.

F. 3100. A sequence (a_n) is defined by recursion

$$a_0=a_1=1,\qquad a_{n+1}=a_n+\frac{1}{a_{n-1}}.$$

Show that $a_n\geqslant\sqrt{n}$.

F. 3101. Solve equation

$$\cos x-\cos 2x+\cos 3x-\cos 4x=\frac{1}{2}.$$

F. 3102. Prove that the integer part of the number $(5+\sqrt{21})^n+1$ is divisible by 2^n.

F. 3103. Every point of a segment AB lies inside an equilateral triangle. The segment is projected orthogonally onto each side of the triangle. In which direction of segment AB does the sum of the lengths of these projections attain its maximum and its minimum values, respectively?

F. 3104. Consider n lattice points selected on a square grid. A segment determined by two of them is called ‘good’ if its midpoint is also a lattice point. Find, in terms of n, the minimum possible number of ‘good’ segments.

F. 3105. A tetrahedron has an inscribed sphere of unit radius. Each plane tangent to the sphere that is parallel to one face of the tetrahedron cuts a small tetrahedron off the original one. Find the sum of the inradii of these tetrahedra.

F. 3106. Derive relation

$$\sin 10^\circ+\sin 30^\circ+\sin 50^\circ+\ldots+\sin 170^\circ=\frac{1}{\sin 10^\circ}.$$

F. 3107. Prove that the number $1991^{1991}+1992^{1992}+\ldots+1996^{1996}$ is not a perfect square.

Submitted by N. Szanyi, Salgótarján

F. 3108. Given the sequence 1, 2, …, 100, our goal is to transform it into the sequence 2, 3, 4, …, 99, 100, 1. At one step we are allowed to transpose two not necessarily neighbouring elements of the sequence. Find the minimum number of steps required to reach our goal.

Submitted by E. Fried Budapest

F. 3109. The opposite sides of hexagon $ABCDEF$ are parallel to each other. Prove that the area of triangle ACE equals that of triangle BDF.

F. 3110. The incircle of triangle ABC touches sides AC and BC at points E and F, respectively. Let G (H) denote the foot of the perpendicular dropped from vertex A (B) to the bisector of $\angle ABC$ $(\angle BAC)$. Show that E, F, G and H are collinear points.

F. 3111. A sphere incident to two neighbouring vertices of a unit cube touches those faces of the cube that do not contain either vertices. Calculate the radius of the sphere.

F. 3112. Find the positive solutions of the system of equations

$$x+y=uv$$
$$xy=u+v$$
$$xyuv=16$$

in unknowns x, y, u, v.

F. 3113. Is it possible that three successive terms of an arithmetic progression of common difference other than 0 form three, not necessarily successive, terms of a geometric progression, in the same order? Can they form three consecutive terms, not necessarily in the same order?

F. 3114. An 8×8 array consists of real numbers. There is 0 written at each corner of the array, and every number in the array is less than or equal to the arithmetic mean of its neighbours in the array. (Two fields of the tableaux are said to be neighbours if they share a common edge.) Reconstruct the array.

F. 3115. In a triangle ABC, $AB=AC$ and $\angle BAC=36^\circ$. The angle bisector of $\angle ABC$ cuts side AC at point D. Let O_1, O_2 and O_3 denote the circumcenters of triangles ABC, BCD and ABD, respectively.
a) Prove that triangle $O_1O_2O_3$ is similar to triangle DBA.
b) Calculate the ratio $O_1O_2:O_2O_3$.

National Mathematical Student Competition, 1995–96

F. 3116. A square of side x is inscribed into a triangle ABC such a way that two of its vertices are incident to side BC, and the other two vertices are lying on sides AB and AC, respectively. Let y and z denote the sides of the squares inscribed into the triangle the other two similar ways, respectively. Derive inequality

$$\frac{a}{x}+\frac{b}{y}+\frac{c}{z}\geqslant 3+2\sqrt{3}.$$

Submitted by I. Varga, Békéscsaba

F. 3117. We are given a solid regular tetrahedron of unit edge. A small regular tetrahedron of edge $1/n$ is cut off the tetrahedron at each of its vertices, where n is an integer greater than 1. Find the value of n for which the surface area and the volume of the truncated tetrahedron has the smallest possible ratio.

National Mathematical Student Competition, 1995–96

F. 3118. Let a_1, a_2, …, a_n denote positive numbers. Derive inequality

$$\prod_{k=1}^{n}\left(1+\frac{1}{a_k}\right)\left(1+\frac{a_k}{k^2}\right)\geqslant (n+1)^2.$$

Submitted by S. Róka, Nyíregyháza

F. 3119. At a volleyball tournament, there were three teams so that the first of them defeated the second one, the second team defeated the third one, and the third team defeated the first one. Prove that there were two teams having the same number of victories.

F. 3120. Does there exist a polynomial $P(x, y)$ of degree at most two that attains the values 1, 2, 3, 4, 5, 6, 7, 8, 10, each of them exactly once on the set $\{1, 2, 3\} \times \{1, 2, 3\}$?

Submitted by Z. Blázsik, Szeged

F. 3121. Given a triangle ABC, extend its sides AB and AC beyond vertex A by a segment of length BC. Extend also sides BC and BA beyond B by a segment of length CA, and, finally, extend sides CA and CB beyond C by a segment of length AB. The far ends of these extensions determine a hexagon. Show that the area of this hexagon is at least 13 times as large as the area of triangle ABC.

F. 3122. The sides of a triangle ABC are denoted by a, b, c. Reflecting the triangle in its incentre, we receive triangle $A_1B_1C_1$. Prove that the perimeter of the hexagon arising as the intersection of triangles ABC and $A_1B_1C_1$ is not greater than $2(ab+bc+ca)/(a+b+c)$.

National Mathematical Student Competition, 1995–96

F. 3123. Let A, B, C, and D be four points in the (three dimensional) space. Derive inequality

$$AC^2+BD^2+AD^2+BC^2 \geqslant AB^2+CD^2.$$

Discuss also the case of equality.

Submitted by S. Róka, Nyíregyháza

F. 3124. Solve equation

$$2^{-\sin^2 x}+2^{-\cos^2 x}=\sin y+\cos y.$$

Submitted by L. Czirok, Szekszárd

F. 3125. Let $0<x,\ y,\ z<b$. Derive inequality

$$\frac{x}{b^2+by+zx}+\frac{y}{b^2+bz+xy}+\frac{z}{b^2+bx+yz}<\frac{1}{b}.$$

Submitted by M. Bencze, Braşov

F. 3126. At the end of a certain table-tennis tournament we found that whichever way the competitors were arranged in a circle, there was at least one competitor who beat both of his or her neighbours. Prove that there can be selected some competitors (at least one, but definitely not all) so that each of them beat every competitor who is not among the selected ones.

F. 3127. A closed disc of unit radius is covered by smaller discs, all of which are of the same size.
a) Find the smallest number of discs in such a covering system.
b) Given that the number of discs is as small as possible, find the smallest possible common radius of the discs.

National Mathematical Student Competition, 1995–96

F. 3128. Let $a \leqslant b \leqslant c$ denote the sides of a triangle. Find all possible values of expression

$$\frac{(a+b+c)^2}{bc}.$$

National Mathematical Student Competition, 1995–96

F. 3129. In a triangle, we are given a point on each side of the triangle, and we are also given three concurrent rays e, f, g that are incident to the three vertices of the triangle, respectively. Construct the triangle, using only a (straight edge) ruler.

Advanced problems:

N. 73. Is there any irrational number c in the interval (0, 1) such that neither c nor $\sqrt{c}$ has a digit 0 in its fractional part?

N. 74. A positive integer is called *varied* if multiplied by any positive integer m the product contains all the 10 possible digits. It is n-*varied* if it has the required property for every integer $1 \leqslant m \leqslant n$.
a) Prove that *varied* numbers do not exist.
b) Does there exist any 1995-*varied* number?

N. 75. Find the maximum ratio of the product of the sides and the product of the diagonals of an inscribed pentagon.

N. 76. For every positive integer n, prove that there exists a polynomial of degree n with integer coefficients of absolute value at most n, which admits 1 as a root with multiplicity of at least $[\sqrt{n}]$.

N. 77. Define a linear order $\prec$ on the set of positive integers so that $a \prec b \prec c$ implies $2b \neq a+c$. The relation $\prec$ is a linear order if
a) every pair a, b satisfies one and only one of the relations $a \prec b$, $b \prec a$ and $a=b$ and
b) $a \prec b$ and $b \prec c$ implies $a \prec c$.

N. 78. Given a circle, construct its center with the help of a single compass.

N. 79. A solution of the system of equations

$$x_1^3+x_2^3+x_3^3=y_1^3+y_2^3+y_3^3,$$
$$x_1+x_2+x_3=y_1+y_2+y_3$$

is called trivial if the triplets x_1, x_2, x_3 and y_1, y_2, y_3 differ only in the order of the three numbers. Show that there exists an integer $n \geqslant 1995$ for which at least 99 per cent of the solutions of (4) are trivial among the numbers $1, 2, \ldots, n$.

N. 80. Define sequences (a_n), (b_n) and (c_n) in the following recursive manner: $a_1=1$, $b_1=2$, $c_1=4$; furthermore, for every integer $n>1$, a_n is the smallest positive integer other than $a_1, \ldots, a_{n-1}, b_1, \ldots, b_{n-1}, c_1, \ldots, c_{n-1}$; b_n is the smallest positive integer other than $a_1, \ldots, a_n, b_1, \ldots, b_{n-1}, c_1, \ldots, c_{n-1}$; and $c_n=n+2b_n-a_n$. Prove that $0<(1+\sqrt{3})n-b_n<2$.

N. 81. Let z denote a complex number of absolute value 1. Prove that there exists a polynomial of degree 1995, all the coefficients of which are +1 or -1, so that $|p(z)| \leqslant 4$.

N. 82. Show that there exists, for infinitely many n, a polynomial of degree n with the following properties: its coefficients are integers, its leading coefficient is less than 3^n, and it has n distinct roots in the interval (0; 1).

N. 83. Is it true that every irrational number has an (integer) multiple among the decimals of which there are either infinitely many '0' digits or infinitely many '9' digits?

N. 84. In the interior of the unit square $ABCD$, two points P and Q are given. Prove inequality

$$13(PA+QC)+14PQ+15(PB+QD)>38.$$

N. 85. We are given an $n\times n$ array of numbers $+1$ and -1 with the following property. Whenever we compare two rows of the array, the number of positions at which they coincide is the same as the number of positions at which they differ from each other. Prove that the sum of the entries of the array is not greater than $n^{3/2}$.

N. 86. We have two sand-glasses that can measure p and q minutes, respectively, where p and q are coprime positive integers. Our intention is to use them to measure n minutes. Initially, all the sand can be found in the lower parts of the glasses. When we start an experiment, and later, when one of our sand-glasses stops, we are allowed to turn upside down either any one or both of the glasses. Show that we can measure the required time, if $n\geqslant pq/2$.

N. 87. We are given a lattice rectangle the sides of which are not necessarily parallel to the coordinate axes. It is partitioned into lattice triangles of area $1/2$. Show that some of these triangles are right-angled ones, in fact, the number of right triangles in the decomposition is at least $2/\sqrt{5}$ times the length of the shorter side of the given rectangle.

N. 88. Is there any continuous function $f:[0,1]\to R$ that attains every element of its range an (finite) even number of times?

N. 89. Let, by definition, a word be any finite sequence of letters. Given a word, we may repeatedly apply the following operations:
a) We may erase either the first or the last letter of the word;
b) We may 'double' the word, i.e. we may concatenate two identical copies of the word.
Is it possible to alter the word $ABCD\dots XYZ$ into the word $ZYX\dots DCBA$ this way?

N. 90. A convex polygon is orthogonally projected onto a plane. Show that its projection can be covered by a congruent copy of the polygon.

N. 91. Prove that for every positive integer n, there exists a polynomial p of degree at most $100n$, satisfying inequality

$$|p(0)|>|p(1)|+|p(2)|+\ldots+|p(n^2)|.$$

N. 92. Let $a_1, a_2, \dots, a_n$ be n different numbers. Derive equality

$$\sum_{i=1}^{n}\left(a_i\cdot\prod_{j\neq i}\frac{1}{a_i-a_j}\right)=0.$$

N. 93. Define the Lucas numbers by recursion

$$L_0=2,\quad L_1=1,\qquad L_{n+2}=L_{n+1}+L_n\qquad (n=0,\,1,\,\dots).$$

Prove that if n is even, then every prime factor of L_n-2 is a divisor of $L_{n+1}-1$, too.

N. 94. Find a positive number z for which $[z^n]-n$ is an even number for every positive integer n.

N. 95. Let $n\geqslant 3$ be an integer. Show that one can partition a square into more than 2^n rectangles so that any line parallel to a side of the square intersects the interior of at most n rectangles. Prove furthermore, that no partition of the square into more than 3^n rectangles has the same property.

N. 96. Prove that there is no arithmetic progression consisting of four different perfect squares.

N. 97. Define a sequence (a_n) by recursion

$$a_0=3,\quad a_1=0,\quad a_2=2,\quad a_{n+3}=a_{n+1}+a_n\qquad (n=0,\ 1,\ 2,\ \ldots).$$

Prove that, for any prime number p, $p\mid a_p$.

N. 98. Let p denote a polynomial with integer coefficients. Prove that, for arbitrary integers m and n, n divides $p(m+n)-p(m)$. Is there any function $p:\mathbf{Z}\to\mathbf{Z}$ that is not a polynomial with integer coefficients, yet has the previous property?

N. 99. Show that, for infinitely many positive integers L, every term of sequence

$$a_0=0,\quad a_{n+1}=\frac{1}{L-a_n}\qquad (n=0,\ 1,\ 2,\ \ldots)$$

can be written as the ratio of two successive Fibonacci numbers.

N. 100. There are given $3k+2$ points in the plane, no three of them are collinear. Prove that there is one point which determines at least $k+1$ different distances with the other points.

N. 101. Prove identity

$$\sum_{j=0}^{r}\binom{d}{j}\binom{d-r-j-1}{r-j}=\frac{2^r}{r!}\prod_{k=0}^{r-1}(d-(1+2k)).$$

N. 102. Choose 1996 straight lines in the plane such a way that any two of them are intersecting and no three of them are concurrent. The lines divide the plane into regions. Find the minimum resp. the maximum number of angular regions that may arise this way.

N. 103. Call a (simple, connected) graph randomly Eulerian if whichever way we walk along its (consecutive) edges, taking care that we visit every edge only once, we visit every edge of the graph sooner or later. Find all randomly Eulerian graphs.

Submitted by D. Varga, Budapest

N. 104. The terms of a sequence (a_n) satisfy relations

$$\sum_{d\mid n}a_d=2^n\qquad (n=1,\ 2,\ \ldots).$$

Prove that $n\mid a_n$ for every positive integer n.

N. 105. We are given lot of different sets, their number is large enough (i.e. greater than a certain sufficiently large number). Prove that among the given sets one can find 1996 sets so that no one of them is obtained as the union of two other sets.

N. 106. In a graph of 1996 vertices that does not contain an isolated vertex (i.e. a vertex of degree zero) we select a subset of vertices at random. Prove that they form a covering system with probability less than 0.51, that is, the probability having an edge in the graph without an endpoint belonging to the selected set of vertices is greater than 0.49.

N. 107. Given a graph, we may walk on the vertices of the graph according to the following rules. In each step we may move from a vertex to one of its neighbours (if there is any) that we have not visited before. Call a graph randomly Hamiltonian if whichever way we walk on its vertices (according to the above described rule), we visit every vertex of the graph sooner or later. Find all randomly Hamiltonian graphs.

Submitted by A. Antos and D. Varga Budapest

N. 108. On the surface of a sphere there is given a closed curve that intersects every great circle of the sphere. Prove that the length of the curve is not less than the circumference of a great circle of the sphere.

THE PROBLEMS OF THE 1995 KÜRSCHÁK CONTEST

1. A lattice rectangle with sides parallel to the coordinate axes is divided into lattice triangles, each of area $1/2$. Prove that the number of right triangles among them is at least twice the length of the shorter side of the rectangle. (A lattice polygon is a polygon all coordinates of the vertices of which are integers.)

2. We are given a polynomial in n indeterminates. Assigning either +1 or -1 to each of its variables, the value of the polynomial is positive (negative) if the number of -1's is even (odd). Prove that the total degree of the polynomial is at least n, i.e. in one of its monomials the sum of the exponents of the indeterminates is at least n.

3. We are given four points A, B, C, D no three of which are collinear. Let E and F denote the intersection points of lines AB and CD resp. lines BC and DA. Construct circles on the segments AC, BD and EF as diameters. Show that either the three circles have a common point or they are mutually disjoint.

Physics

Experimental problems

172. Let us measure the internal volume of a beer bottle, as well as the volume of it's glass container. Let's also measure the density of the container. What would be the minimal amount of water required to make the bottle float in water in a stable manner? How much additional water would make it sink?

Budó Ágoston Physics Competition, Békéscsaba

173. Measure the moment of inertia of the front wheel of a bicycle with respect to its axe of rotation.

Submitted by M. Molnár, Szeged

174. After the rain raindrops gather on the level line covered with plastic at roughly equal distances from each other. Measure whether there is any connection between the distance of the raindrops forming on a thin line and the diameter of the line. Repeat the measurement with another liquid instead of water.

Submitted by Zs. Rajkovits, Budapest

175. Look at a point-like light source through different woven textiles. Describe what you see. What is the explanation of the phenomenon?

Submitted by G. Tichy, Budapest

176. Prepare a solution of salt in water that does not freeze even in the deep-freezer (there is neither solid salt nor ice in it). Measure between what boundaries can the ratio of the mass of the salt to that of the water fall at the temperature of the deep-freezer.

Submitted by Gy. Radnai, Budapest

177. Measure the refractive index of ice.

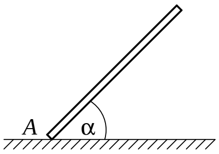
Submitted by A. Részegh, Budapest

178. Attach a body to the bottom of another body hanging on a long thread, by means of a rubber thread. Observe the conservation of angular momentum with the help of this instrument.

Submitted by M. Gündischné Gajzágó, Békéscsaba

179. Measure the internal resistance and the utilizable energy of three 1.5 V galvanic cells of different types. What measurable electric quantity does the price of the cell depend on?

Submitted by Gy. Radnai, Budapest



180

180. A long, straight ruler is released from the position shown in the *figure.* What is the displacement in cm of point A of the ruler by the time it comes to rest again? Measure the displacement as a function of the angle α for at least three different values of the friction coefficient μ.

Submitted by I. Varga, Békéscsaba

Exercises, problems and advanced problems

FGy. stands for the exercises (for 14–16 year-old students), FF. stands for the problems (for 17–18 year-old students), and FN. for advanced problems in physics.

FGy. 2904. Consider 2 steel balls of identical diameter, one solid, one hollow. How would you decide which one is hollow in a situation of zero gravity (for example in a satellite circling the Earth)?

Öveges József Physics Competition for primary schools

FGy. 2905. We would assemble 3 identical cylindrical magnets that have been magnetized in the direction of their longitudinal axles. If we try to pull the assembly apart by their outer components, where would the break occur if the axle is
a) vertical, b) horizontal?

Öveges József Physics Competition for primary schools

FGy. 2906. Trains of the Tokaido Express run regularly on half hourly basis. The duration of each trip is 3 hours. (I.e. at the exact moment of departure another train would be arriving). How many oncoming trains would each train meet during it's run?

Lánczos Kornél Physics Competition, Székesfehérvár

FGy. 2907. Leonardo da Vinci (1452–1519) believed that the time required for an object to slide down an inclined plane without friction is proportional to the length of the plane, if the elevations are identical. Was his assumption correct?

Holló András Physics Competition, Jászberény

FGy. 2908. The capsule of a toy rocket ejects 10 g fuel, with a velocity of 10 m/s relative to the capsule. The rocket is put into motion on a horizontal frictionless cable, it's motion is resisted by a drag-force proportional to the square of it's velocity. The drag force is 0.004 N at the velocity of 1 m/s. What maximal speed can be reached by the rocket?

Mikola Sándor Physics Competition

FF. 2909. The decision reached by a town council was to declare smog alarm if the atmospheric carbon monoxide content has reached $4\cdot10^{-5}$ kg/m^3.

a) How many CO molecules would be a man take in with a single breath of 500 cm^3?

b) What is the average energy of a CO molecule in the lung at 37°?

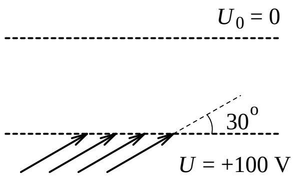
FF. 2910. A car follows a truck on a perfectly straight road, both vehicles travelling at 72 km/h speed. What minimal following distance should be observed by the car in order to avoid flying stones from the rear wheels of the truck?

Nagy László Physics Competition, Kazincbarcika

FF. 2911. We would lower a 76 cm long glass tube of open ends into mercury. We would slowly lift the tube out, holding one end closed with a finger. How high will be the remaining mercury column in the tube? What will happen if the tube is tilted?

Vermes Miklós Physics Competition, Sopron

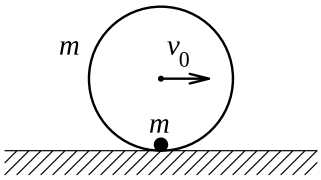
FF. 2912. A plane capacitor armatures are formed by two grids. The voltage between the armatures is 100 V. An electron beam reaches the positively charged armature at 30 degrees.



2912

a) What would be the maximum kinetic energy of the electrons which can not reach the negatively charged armature?

b) What is the terminal velocity of the electrons whose kinetic energy is twice that of the minimal energy needed for penetration?



2913

FF. 2913. Onto the hoop of $m=1.6$ kg mass and radius $R=0.2$ m positioned in a vertical plane, another object of mass m point load is fastened. In the beginning the load is in the lowest position. From this position the hoop is rolled away horizontally with a starting velocity of v_0. The hoop rolls clearly.

a) What should be the maximum starting speed of the centre of this hoop to keep the rolling hoop constantly in touch with the ground.

b) What force is exerted on the ground by the hoop when the point load is R distance above the ground?

Párkányi László Physics Competition, Pécs

FGy. 2914. When does the wooden boat sink deeper: if it leaks and the water flows into it through the hole or if the intact boat is scooped full with water?

Submitted by G. Takács, Budapest

FGy. 2915. Two worms of equal mass 20 g climb a 10 cm high, very thin wall. One of them is thin and 20 cm long, the other one is fat and 10 cm long. Which one has to do more work "against gravity" till they cross half of the wall? What is the ratio of the work done by the two worms?

Submitted by L. Holics, Budapest

FGy. 2916. What is the pressure inside an air bubble of diameter 1 μm, 5 meters under water?

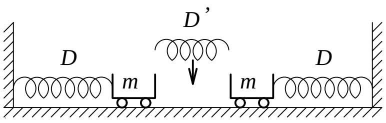
Slovakian physics competition problem

FGy. 2917. What can a gas be if the average thermal speed of its molecules increases from 400 m/s to 500 m/s when its temperature is increased by 100 K?

Rumanian physics competition problem

FF. 2918. What is the minimum magnitude and the direction of the force that has to be exerted by a worker of weight G_1 to push a box of weight G_2? The static friction coefficient is μ_1 between the shoe of the worker and the floor and μ_2 between the box and the floor. E.g. let the mass of the box be 150 kg, that of the worker 70 kg, $\mu_1=\mu_2=0.4$. What is the situation if the worker puts a pair of more suitable shoes on, with which μ_1 can be increased to 0.6 or 0.7?

Submitted by F. Woynarovich, Budapest



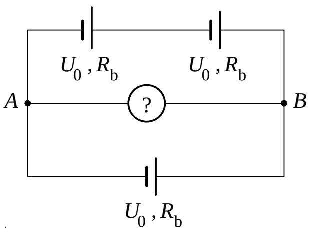
2919

FF. 2919. The little carriages shown in the figure can oscillate without friction, with frequency f_0. A spring of force constant $D'<2D$ is attached to the two carriages in such a way that neither of the three springs be stretched.

a) What is the frequency of the oscillations of this coupled system if the carriages are moved by the same distance and into the same direction, or by the same distance but into opposite directions from their original positions, and then left alone?

b) How does the difference f_2-f_1 depend on the strength of the coupling? Make a graph of (f_2-f_1) versus D'.

Submitted by M. Gündischné Gajzágó, Hatvan



2921

FF. 2920. How high should a thread pendulum be lifted at the Equator so that its period increases by 1 per thousand?

Submitted by L. Keglevich, Budapest

FF. 2921. The circuit shown in the diagram is made with three identical batteries. What is the reading on the

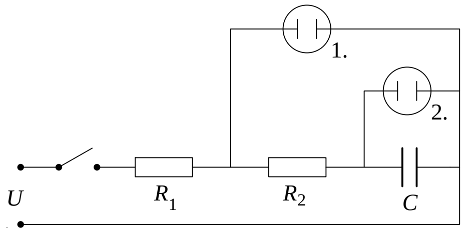
a) ideal ammeter, b) ideal voltmeter

connected between points A and B?

Submitted by I. Varga, Békéscsaba

FF. 2922. The switch is suddenly closed in the circuit shown in the diagram.
a) Make a sketch of the changing of the p.d. on discharge tubes 1. and 2. in time.
b) Which tube lights up first if their sparking voltage is 110 V?
c) Make an estimate of the time elapsed before the first flashing.
Data: $U = 200$ V, $R_1 = 20$ kΩ, $R_2 = 20$ kΩ, $C = 50$ μF.

Submitted by L. Madas, Budapest



2922

FN. 2923. An ideal gas made up by thirty helium and thirty neon atoms is in equilibrium in a vessel.
a) What is the probability in percentage that there are exactly thirty atoms in half of the volume of the vessel?
b) What is the probability in percentage that half of these are helium atoms?
c) How sure a statement is it that the number of the atoms in half of the volume is between 20 and 40?
d) Which of the following statements is more likely:
"The number of atoms in half of the volume does not differ from 30 by more than 2."
"The number of atoms in half of the volume differs from 30 by at least 3."
e) Suppose that originally the helium atoms were in one half of the vessel and the neon atoms were in the other one, and then they mixed spontaneously. By how much has the entropy increased?

Submitted by Gy. Radnai, Budapest

FGy. 2924. A piece of 0 °C ice of unknown mass is placed into a filter on the orifice of a vessel. The ice is melted with the help of 15 °C water flowing from the tap. 10 cm^3 of water are poured each second evenly onto the ice for a period of 9 minutes and 10 seconds. The ice melts and the temperature of the water gathered in the vessel is 7 °C. What was the mass of the piece of ice?

Submitted by G. Takács, Budapest

FGy. 2925. Only 5 % of the volume of an egg floating in salted water emerges from the water. If pure water is cautiously layered on top of the salted water, half of the egg will be in salted water and half in pure one.
a) What is the density of the egg and of the salted water?
b) At least how much pure water has been layered on top of the salted water if the egg sank when the water was mixed?

Submitted by M. Gündischné Gajzágó, Hatvan

FGy. 2926. Is it conceivable that tin can be melted by means of water?

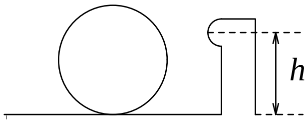
Submitted by Gy. Radnai, Budapest

FGy. 2927. A spherical celestial body of average density ϱ revolves around its own axe. What angular velocity is needed to assure weightlessness on the equator of the celestial body?

Submitted by P. Várkonyi, Budapest

FF. 2928. We wanted to make a thread pendulum with a period of 2 seconds from thin fisher string. Unfortunately the string stretched by 4 % right in the first 12 minutes. How many periods did the pendulum do during this time if the stretching occurred evenly?

Submitted by I. Varga, Békéscsaba



2929

FF. 2929. The edge of the billiard tables is not vertical but of the shape shown in the figure. What would happen to the balls rolling against the wall if the edge of the table were vertical? Why is it good if the height h is 0.7 times the diameter of the ball? (Assume that the edge of the table exerts a level force onto the ball when they collide.)

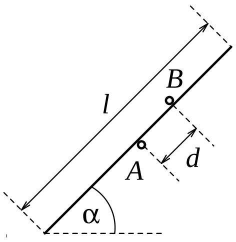
Submitted by P. Vankó, Budapest

FF. 2930. The radii of three concentric metal spheres with thin walls are $r_1 = 1$ cm, $r_2 = 10$ cm and $r_3 = 20$ cm. The charges of the spheres are respectively: $Q_1 = 2 \cdot 10^{-8}$ C, $Q_2 = Q_3 = -2 \cdot 10^{-8}$ C.
a) What is the total energy of the electric field of this charged system?
b) What is the potential of the innermost sphere if the potential is zero at infinity?

Submitted by L. Holics, Budapest

FF. 2931. An Opel and a Skoda approach each other. The speed of the Opel is 90 km/h, that of the Skoda is 54 km/h. A sound emitted by the Skoda is perceived by the driver of the Opel at a frequency 127.3 Hz higher when approaching each other than when moving away from each other. (The speed of sound in motionless air is 340 m/s.) What is the frequency of the emitted sound? What are the perceived frequencies? How do the results change if a wind of 36 km/h blows against the Opel?

Submitted by L. Madas, Budapest



2932

FF. 2932. Two thin nails (A and B) are driven into the vertical wall and a thin, rigid wooden lath of homogeneous mass distribution is slipped between them as shown in the picture. In the beginning the middle of the lath is right above A and the lath is released in this position. The distance between the two nails is $d = 10$ cm, the length of the lath is $l = 100$ cm, the angle between the lath and the horizontal plane is $\alpha = 45°$, the dynamic friction coefficient between the lath and the nails is $\mu = 0.25$.
a) What will the motion of the lath be like?
b) What will be the highest speed of the lath?
c) For how long will the lath move?

Submitted by I. Varga, Békéscsaba

FN. 2933. The experiment described in the previous problem is repeated with an iron rod instead of the wooden lath, for which $\mu = 0.1$. Estimate the r.p.m. of the rod when it falls freely. Give the order of magnitude of the accuracy of the estimate.

Submitted by P. Gnädig, Budapest

FGy. 2934. It can be read in the physics notes of Bolyai Farkas (1755–1856) that "…Some bodies can be split into very many pieces; it is an old piece of knowledge that a horse with its rider can be covered with one gram of gold. . .” Estimate how many atomic layers would constitute the covering foil of the real size statue of a mounted man if 1 gram of gold were used.

Submitted by M. Gündischné Gajzágó, Hatvan

FGy. 2935. We have to arrive at a certain remote city by car by a fixed time. We have calculated that we have to travel at an average speed of at least 60 km/h. There is however a traffic jam on the road, so we only cover one fifth of the distance in a quarter of the available time. What minimum average speed do we have to keep for the rest of the journey to arrive in time?

Submitted by G. Takács, Budapest

FGy. 2936. A vehicle travelling at a high speed has to stop on a straight road section. It has two braking modes. By which one shall the driver start braking — the stronger or the weaker braking mode — in order to stop in a distance as short as possible?

Submitted by I. Varga, Békéscsaba

FGy. 2937. How high should a parabolic antenna be fixed at the North Pole in order to receive the emissions of a satellite on a synchronous orbit around the Earth?

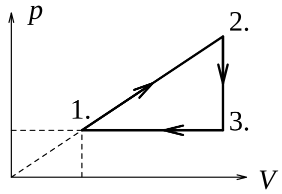
Submitted by G. Petróczi, Kazincbarcika

FF. 2938. The temperature in a crowded concert hall increases from 20 °C to 26 °C. How much does this change the frequencies of the tin whistles of the organ tuned at 20 °C? Compare the magnitude of the effect of dilatation to that of the changing of the speed of sound.

Submitted by Gy. Honyek, Budapest

FF. 2939. A certain quantity of monatomic ideal gas makes the cycle shown in the figure. Due to the structure of the equipment, the ratio of the highest to the lowest temperature of the gas, measured on the Kelvin scale, cannot be greater than 16. What can be the highest thermal efficiency of such a cycle?

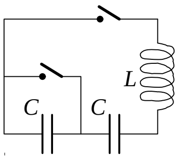
Submitted by L. Kotek, Pécs
(Mikola Sándor Physics Competition, Sopron)



2939

FF. 2940. An ant drags a stalk of straw across a cylindrical branch of wood lying on the ground, moving away at a steady speed v from the branch with the end of the straw. The stalk of straw is three times longer than the diameter of the branch and moves in a plane perpendicular to the branch. What is the kinetic energy of the straw when its middle point touches the branch?

Submitted by I. Légrádi, Sopron



2941

FF. 2941. Two capacitors of capacitance C are connected is series and charged to a voltage U. A coil of inductance L is connected to this “battery of capacitors” of voltage U, and one of the capacitors is short circuited after a time lapse τ. What will be the maximum value of the charge of the other capacitor after this?

Submitted by L. Király, Veszprém
(*Bródy Imre Physics Competition,* Ajka)

FF. 2942. The image of a vertical slot is focused by a lens onto a screen. The screen is parallel to the slot, the magnification is double. The slot is then tilted in the vertical plane towards the screen by 45°. By how many degrees has the screen to be tilted so that the image be again focused onto it?

Submitted by Gy. Radnai, Budapest

FN. 2943. A thin copper ring is placed into a homogeneous magnetic field of changing induction, perpendicular to its plane. The induction starts to increase at $t=0$ and increases evenly from zero to 2T in 0.2 s. At what constant angular velocity ω does the ring have to be turned so that there be no elastic stress in it at $t_1=0.1$ s. Can the problem be solved in the case of a magnetic induction decreasing from 2T to 0?

Submitted by L. Holics, Budapest

FGy. 2944. A ship is anchored in a harbour where the depth of the water is 10 m. A stormy wind is blowing, pushing the ship with an even, nearly level force of 7000 N. The rope of the anchor is 50 m long and stretched. What is the force exerted by the rope? At least what work does the crew do when they pull in 30 m of the rope? What will be the force stretching the rope afterwards?

Budó Ágoston Physics Competition, Szeged

FGy. 2945. Two cyclists start at the same time towards each other from two villages. When they meet, the first cyclist increases his speed to 1.5 times the original value, the second to 1.2. Thus it takes half as long to the first cyclist to cover the distance between the two villages as to the second. At what portion of the way do they meet?

Submitted by G. Zs. Tóth, Budapest

FGy. 2946. The air near the ceiling of a closed, heated room is warmer than near the floor. Where is the absolute vapour content (the mass of water vapour in unit volume) greater: downwards or upwards?

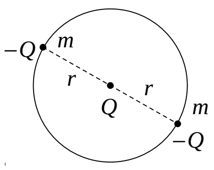
Submitted by I. Varga, Békéscsaba

FGy. 2947. We know from Gay-Lussac's first law that the thermal dilatation coefficient of gases at constant pressure is $1/273\ °\mathrm{C}^{-1}$ at 0 °C. What is their thermal dilatation coefficient at a temperature of -23 °C?

Submitted by P. Simon, Pécs

FF. 2948. Imagine that London is joined with the North Pole by an evacuated, straight tunnel in which a magnet floated train travels with no engine. How long does it take to the train to get from London to the North Pole? What is its maximum speed during the journey? (Consider the Earth to be a homogeneous sphere.)

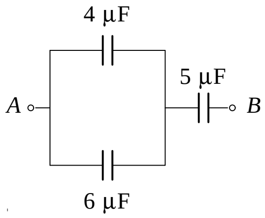
Submitted by Z. Kovács, Kolozsvár



2949

FF. 2949. Two bodies of charge $-Q$ each orbit a body at rest of charge Q on a circular orbit of radius r. What is the angular velocity of the orbiting bodies? What is the energy of the system?

Submitted by L. Kotek, Pécs



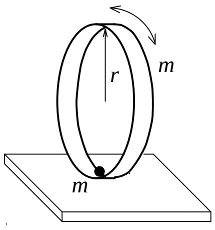
2950

FF. 2950. The system of capacitors shown in the diagram is charged with the help of a voltage of 300 V connected between points A and B. The power supply is then removed and the capacitor of 6 μF is exchanged with that of 5 μF without discharging them and without caring for the polarities. What p.d. can occur between points A and B?

Submitted by F. Schuszter, Budapest

FF. 2951. How many lines does a 2 cm large optical grating consist of if the first order spectrum seen on the screen placed at a distance of 1.5 m from it measures 3 cm from the blue ($\lambda_1 = 400$ nm) to the red ($\lambda_2 = 700$ nm)? What happens if half of the grating is covered?

Submitted by L. Keglevich, Budapest



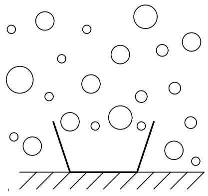
2952

FN. 2952. A point-like body of mass m is fixed to a homogeneous hoop of mass m and radius r. The system is steady in a stable equilibrium position on a level plane. The hoop is then turned by a small angle around its axe and left alone. What will the period of the occurring motion be if

a) friction is negligible,

b) static friction is large enough and the hoop makes a pure rolling motion?

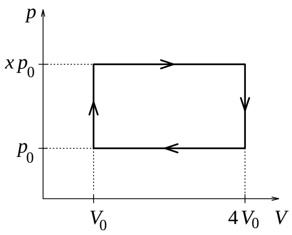
Submitted by E. Szvetnik, Budapest



2953

FGy. 2953. During an experimental demonstration, soap bubbles were blown above a glass tureen. The bubbles falling beside the tureen blew up when reaching the table, but those that arrived above the tureen did not fall into it but 'stopped' somewhere at the top of the tureen and blew up there after some time. What preparations did the demonstrator make to evoke this surprising phenomenon?

Submitted by J. Vida, Eger



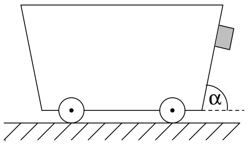
2955

FGy. 2954. Katie invites her class-mates for an afternoon tea. She assigns 10 forints to make tea. How many 2.5 dl mugs can she fill with boiling hot water if a loss of heat of 32 % occurs and the electric energy costs 9.50 Ft/kWh?

Submitted by L. Keglevich, Budapest

FGy. 2955. A monatomic ideal gas makes the cycle shown in the figure. What is the value of x if the thermal efficiency is 25%? Make a graph of the efficiency versus x.

Submitted by L. Kotek, Pécs



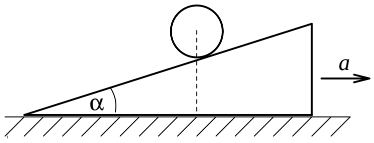
2956

FGy. 2956. The front of a car is at an angle $\alpha = 80°$ to the horizontal plane. A small body is pressed onto this surface as shown in the figure; the small body is then released and the car starts accelerating. What is the acceleration of the car if the small body keeps sticking to the car? The static friction coefficient is $\mu_0 = 1.5$.

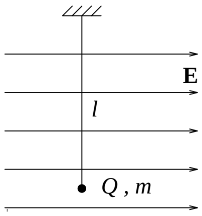
Submitted by Cs. Bordy, Bonyhád

FF. 2957. A sphere is placed onto a slope of inclination α and left alone with no initial velocity. The sphere does not slip and touches the slope all along its motion. What horizontal acceleration should the slope have so that the centre of the sphere arrives at the bottom of the slope on a vertical, straight path?

Submitted by L. Holics, Budapest



2957



2958

FF. 2958. A small body of mass m and charge Q is hung onto an insulating thread of length l and negligible mass as shown in the figure. A homogeneous, horizontal electric field of field strength **E** is suddenly created. Calculate a) the maximum speed reached by the body; b) the maximum deviance of the thread. (Let $Q = 10^{-8}$ C, $m = 2 \cdot 10^{-3}$ kg, $E = 10^5$ V/m.)

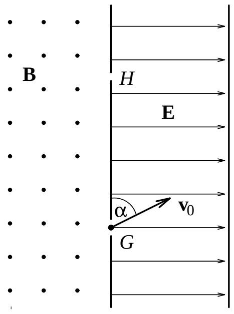
Submitted by I. Varga, Békéscsaba

FF. 2959. Coils are hidden in a 'black box'. The following inductances can be measured at the outputs A, B and C: $L_{AB} = 300$ mH, $L_{BC} = 300$ mH and $L_{AC} = 1.2$ H. What is in the box?

Submitted by J. Lazányi, Debrecen

FF. 2960. A homogeneous electric field of field strength **E** and a homogeneous magnetic field of inductance **B** are perpendicular to each other. Electrons are shot into the electric field from point G shown in the figure, at angle α and with initial speed v_0, so that they cross point H and the magnetic field and return to point G. What relation can be formulated between E, B, v_0 and α? In which field do the electrons spend more time if $\alpha = 60°$?

Submitted by L. Csonka, Vámosszabadi



2960

FF. 2961. What is the rate of transformation of hydrogen into helium in the Sun? How many times is the energy provided by the fusion of hydrogen larger than that provided by the fission of the same mass of $^{235}_{92}$U?

Submitted by L. Madas, Budapest

FN. 2962. A large, spherical space probe painted in black moves in the far space. There is a working computer, powered by a radioactive energy source, inside the probe that makes the temperature of the surface (and the inside) of the probe be 250 K.
a) Does the temperature of the probe change (and if so, how?) if it is surrounded by a spherical, black casing? (The casing is fixed at a small, even distance from the surface of

the probe, by means of a heat insulating framework. There is vacuum between the two surfaces.)
b) What happens if 8 concentric cases are used instead of 1?

Russian physics competition problem

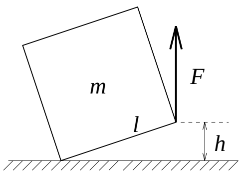
FGy. 2963. A ship put out to sea half an hour ago on a straight path, with a speed of 36 km/h. A passenger missed it and is leaving now by sea taxi. The taxi travels at a speed of 45 km/h, and its whole fare for the way to the ship and back is 0.50 \$/km. The taxi can also travel at a speed of 60 km/h, but then the way towards the ship costs 1.00 \$/km. Which solution is the cheaper one? *Submitted by Z. Édes,* Tata

FGy. 2964. In a forest that covers an area of 5 km^2, the average area occupied by a tree is 8 m^2. Each tree evaporates 10 kg of water per day. How many cubic meters of water does the forest evaporate in a year? How much natural gas should be burnt in order to evaporate the same amount of 20 °C water at its boiling point if this happens with an efficiency of 30 %? (The combustion heat of natural gas is 33 000 J/kg, $c_{\text{water}} =$ 4.2 kJ/(kg °C), the boiling heat of water is 2260 kJ/kg.)

A problem of *Lajos Kiss* (1939–1995)

FGy. 2965. We want to make a cube having a homogeneous mass distribution, mass m and edges of length l, resting on a level table tip over its edge. We lift the cube by a vertical force acting on the center of one of its edges.
a) Calculate and make a graph of the lifting force versus the height h.
b) What work do we do while slowly making the cube tip?

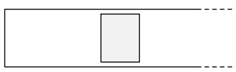
Submitted by Cs. Bordy, Bonyhád



2965

FGy. 2966. “Walking one evening in the garden, the falling of an apple made Newton ask whether the apple would fall down if the tree reached the Moon. And so why does the Moon not fall down?” A problem of *Farkas Bolyai* (1775–1856)

FGy. 2967. A fixed, level cylinder closed by a fixed wall at one end and a piston of mass 20 kg moving without friction at the other end contains a monatomic ideal gas. The piston is initially fixed, then it is released and the enclosed gas is heated in such a way that the piston move with a constant acceleration. The external air pressure is negligible.
a) How much heat has to be given to the gas until the piston reaches a speed of 0.8 m/s?
b) What is the efficiency of the transfer of energy if the aim is to accelerate the piston?


2967

Submitted by L. Kotek, Pécs

FF. 2968. A thin rod of length l and mass m is joined to an evenly rotating vertical axle by means of a hinge of horizontal axle. During the rotation the rod is at an angle α to the vertical axle.
a) What is the angular velocity of the rotation?
b) What is the torque exerted by one half of the rod onto the other one with respect to the center point of the rod? *Submitted by L. Gálfi,* Budapest

FF. 2969. Give an estimate of the mass of the tungsten filament of a 220 V, 100 W lamp. The available data: the specific resistance of the filament (at its working temperature) is $9\cdot10^{-5}\ \Omega\,\mathrm{cm}$, its surface power density is $152\ \mathrm{W/cm^2}$, and its density is $19\cdot10^3\ \mathrm{kg/m^3}$.

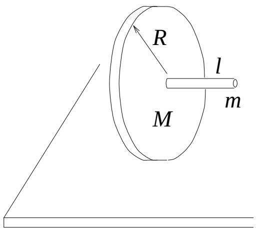
Submitted by S. Mihalovics, Esztergom

FF. 2970. During photo-effect the departure work of potassium is 2.22 eV. What is the frequency of a photon if the electron pushed out of the potassium by it deviates by 2.84 cm when travelling a distance of 10 cm through an electric field of field strength 10 V/m perpendicular to the velocity of the electron? A photon of what frequency would be needed to push out such an electron from the potassium that it would follow a circular path of a radius of 2.17 cm in a magnetic field of inductance 0.01 T?

Submitted by L. Madas, Budapest

FF. 2971. Protons of negligible speed are accelerated by a p.d. of 5 MV. The accelerated protons collide elastically with the atoms of an unknown chemical element. The protons that show a deviation of 90° from their original direction can be slowed down by means of a voltage of 4.23 MV.
a) What can the unknown chemical element be?
b) What is the angle between the velocity of the pushed atomic nucleus and that of the proton after their collision?

Submitted by I. Varga, Békéscsaba



2972

FN. 2972. The "half shafted wheel" composed of a homogeneous disk of mass M and radius R, and of a homogeneous rod of mass m and length l shown in the *figure* rolls purely in such a way that its half shaft always remains horizontal. Show that the wheel rolls along a circular path. What is the radius of this circle if the wheel takes time T to make a whole round? What is the minimum static friction coefficient?

Submitted by P. Balogh, Budapest

FGy. 2973. Do the smaller or the larger raindrops fall faster?

Submitted by S. Mihalovics, Esztergom

FGy. 2974. The whole motion of an iron ball of a diameter of 2 cm falling freely from a height of 5 m, lit by a stroboscope (a light flashing at even time intervals) is photographed. The ball can be seen at 51 places in the developed picture. How many 'separate' balls are there in the picture?

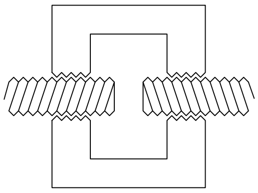
Submitted by I. Varga, Békéscsaba

FGy. 2975. The mass of the weight valve of a pressure cooker is 40 g. The original diameter of the tube leading to it is partly choked at the point where the tube joins the valve. What is the diameter of the narrowed tube if the melting safety valve on the cover of the cooker melted at 125 °C?

Öveges József Physics Competition for secondary schools, Tata

FGy. 2976. A cable tightening device composed of a right threaded and a left threaded screw can be tightened with torque M_1 and loosened with torque M_2. What is the force stretching the cable and what is the friction coefficient between the nuts and the respective screws if the diameter of the screws is d and the distance between two neighboring threads is h? (The thread can be regarded as a slope wounded around a cylinder.) Data: $M_1 = 120$ Nm, $M_2 = 80$ Nm, $d = 12$ mm, $h = 1$ mm.

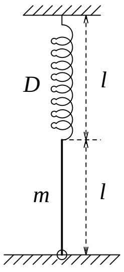
Submitted by A. Gnädig, Budapest



2976

FF. 2977. The stick of mass m shown in the *figure* can turn without friction around the horizontal axle crossing C. What is the initial extension of the spring of force constant D and negligible mass if the stick is in stable equilibrium in the vertical position shown in the figure?

Submitted by P. Balogh, Budapest



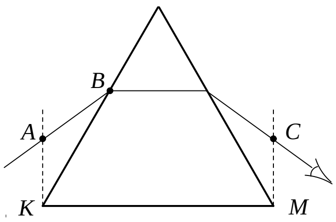
2977

FF. 2978. The central radius of a toroid coil is 10 cm and the diameter of one turn is 12 cm. A current of 20 A flows in the toroid, the number of turns of the coil is 1000. What is the magnetic field formed inside the toroid like? Give the magnitude of the magnetic induction vector as a function of the position.

Submitted by L. Holics, Budapest

FF. 2979. Three pins are stuck in points A, B and C near an optical prism made of a glass of a refractive index of 1.5, whose base is a regular triangle with 8 cm long sides. The pins are stuck parallel to the refracting edge at point L, in such a way that they be seen through the prism to be in a straight line. Point B is in the middle point of KL, while points A and C are on the straight lines perpendicular to KM, at equal distances from KM. If the distance AK is augmented by 2 mm, how should the distance CM be changed so that the three pins be again in a line as seen through the prism?

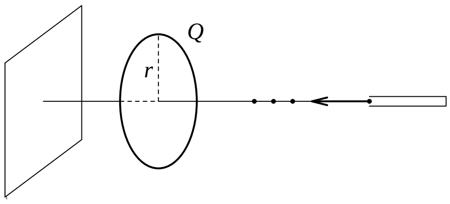
Submitted by M. Gündischné Gajzágó, Hatvan



2979

FF. 2980. In the course of a nuclear physics experiment particles of identical mass and of a charge of $3.2 \cdot 10^{-19}$ C can be shot from a particle gun. The particle gun is situated on the axe of symmetry of a thin metal ring of radius $r = 10$ cm and charge $Q = 10^{-6}$ C, at a distance of 2 m from the center of the ring. A fluorescent screen behind the ring detects the incident particles. According to the accomplished experiments the particles only reach the screen if their initial speed is greater than $2.87 \cdot 10^6$ m/s. (The whole apparatus is in vacuum.) Determine the mass of the particles.

Submitted by L. Kotek, Pécs



2980

FN. 2981. The last April (!) issue of an American review published a sensational announcement about the possible solution of the mystery of the Bermuda triangle. The cosmonauts of the space station SPACETREK continuously measured their orbit by means of a special radar altimeter. During the subsequent analysis of the data, light was thrown on a surprising fact: the surface of the water in the Bermuda triangle is 25 meters lower than the normal sea level of the surrounding ocean! It is presumable that this dip is caused by the presence of a spherical underwater cavity filled with water, formed in the ocean-bottom. There is a layer of water of an average thickness of 6 km above it. Estimate the radius of the cavity using that the average density of the rocks of the ocean-bottom is three times that of water.

Submitted by Gy. Honyek, Budapest

FGy. 2982. Medical specialists examine the cavity of mouth and the throat of the patient with the help of a circular mirror of 10 cm of diameter that has an also circular hole of approximately 12 mm of diameter in the centre. How does this device work?

Submitted by G. Takács, Budapest

FGy. 2983. Try to estimate how much it costs more to make the journey Berlin–Budapest by car with the lights on than with the lights off. (The power of the bulbs is 2×50 W, the average speed is 60 km/h, the efficiency of the engine is 25 %.)

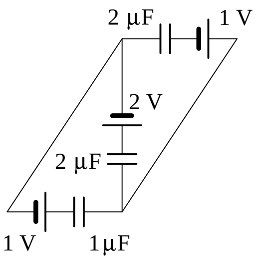
Submitted by Z. Mónus, Tiszaföldvár

FGy. 2984. The 4 dkg plastic stopper of a bottle of champaign reaches as far as 3 cm into the bottle. Taking the bottle out of the refrigerator and leaving it on the garden table the stopper suddenly shoots out with a big bang and rises to a height of 5 m. Estimate the work done by the gas in the bottle on the stopper. A force of at least 20 N is necessary to move the stopper pushed to a depth of 3 cm into the empty bottle.

Submitted by P. Simon, Pécs

FGy. 2985. What is the surface formed by the top points of the parabolic paths of stones thrown from the same point and with identical magnitudes but different directions of their initial velocities?

Submitted by F. Simon, Zalaegerszeg



2987

FGy. 2986. A billiard ball is rolling purely on a level table in a direction perpendicular to a vertical wall. The friction coefficient between the ball and the wall is μ, rolling friction and gravity can be neglected. In what angle will the ball bounce away from the wall? (The collision number is 1, i.e. the momentum perpendicular to the wall changes to -1 times its original value.)

Russian physics competition problem

FF. 2987. To what voltage will each capacitor be charged in the circuit shown in the *figure*?

Submitted by M. Molnár, Szeged

FF. 2988. In the course of a demonstration experiment a coil with an (extraordinarily large) inductance of 600 H and a resistance of 300 Ω is connected to a 12 V accumulator.

What is the initial rate of the increase of the current flowing through the coil? What is the rate of the increase of the current when the rate of the increase of the magnetic energy in the coil is the highest? What is the magnitude of the current then? (The problem can be solved by elementary methods!)

Submitted by Gy. Radnai, Budapest

FF. 2989. A gold fish is swimming in a level plane along the wall of a spherical aquarium of a diameter of 60 cm. We observe the fish from a distance of 30 cm in the same plane. How many times does the fish look bigger when it is at the farthest point from our eye than when it is at the closest to it? How does the angular magnification depend on the distance of our eye from the aquarium? (Assume that the length of the fish is small compared to the size of the aquarium and that the thickness of the wall of the aquarium is negligible.)

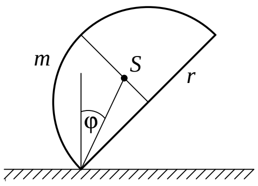
Submitted by P. Balogh, Budapest

FF. 2990. A perfectly reflective surface is lit with a light of a wavelength of 400 nm. The pressure of the radiation is $4\cdot 10^{-8}$ $\mathrm{N/m^2}$. At least how many photons reach a surface of 1 cm^2 per second?

Submitted by L. Madas, Budapest

FN. 2991. A hemisphere of radius r, mass m and homogeneous mass distribution is placed onto its 'edge' on a frictionless level plane in such a way that its centre of mass S be right above the supporting point. It is then released with no initial speed so that it falls onto its plane side. Calculate and plot the force exerted by the supporting point and the angular velocity and the angular acceleration of the hemisphere as a function of the angle φ shown in the *figure.*

Submitted by I. Légrádi, Sopron



2991

International Conference of Young Scientists, Visegrád '96

After previous successful conferences the last International Conference of Young Scientists was held in Visegrád, Hungary, in 1996, in the year of 1100th anniversary of the Conquest of the Carpathian Basin. This old, historical town which played a significant role in the history of our country, is situated on the bank of the Danube in a beautiful environment.

We are convinced that nowadays, when the popularity of sciences decreases permanently, a scientific event which attracts the young generation has special importance.

The conference was organized by the Eötvös Loránd University (Budapest), the Belarus State University (Minsk) and by the Hungarian Physical Society. This conference was a special type of competition in physics, mathematics, computer science and ecology for gifted secondary school students. Every participant had prepared a research report on a subject chosen by himself/herself from any part of the above mentioned sciences. The language of these ten-minute reports was English.

In this year 86 reports were presented by students from 9 countries and the presentations were evaluated and rewarded by an international jury.

Perhaps most of the participating students from secondary schools will later become students at various universities and then researchers reporting on scientific conferences, this first appearance may be a decisive factor in their future scientific career.

To show the character of the competition let us consider a report from one of the most interesting subjects of the non-linear phenomena which was presented by two Hungarian secondary school students.

Zsuzsanna Rajkovits
Chairperson of the Conference, Eötvös University, Budapest

The Saline Water Oscillator

When a cup with a small orifice (ca. 1 mm in diameter) in its base, filled with saline water, is placed within an outer vessel containing pure water, an oscillatory exchange of liquids is generated as salt water flows down and pure water flows up in distinct phases. This phenomenon is known as the saline oscillator. We have investigated the mechanism of the oscillator as well as the dependence of the period of the oscillation on various parameters.

The experimental setup (Fig.1) is rather simple. The cup is fixed with a clamp and the inner solution is dyed with ink for better visibility. The period of the oscillation

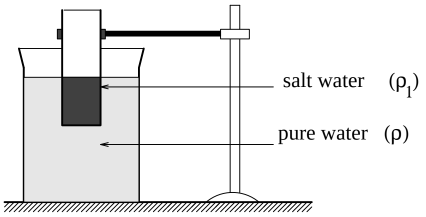


Fig. 1

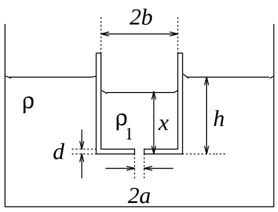


Fig. 2

ranges between a few seconds and several minutes depending on several parameters (see Fig. 2), such as the density of the NaCl solution (ϱ_1), the distance of the base of the cup from the level of water in the outer vessel, or height (h), the radius (a) and length (d) of the orifice, the surface areas of the inner and outer vessels etc. A typical example was 8.9 seconds with $a = 0.5$ mm, $b = 23$ mm, $h = 30$ mm, $\varrho_1 = 1.13$ g/cm^3.

How the oscillator works

The main driving force of the oscillation is not osmotic pressure, as can be demonstrated by using oils of different densities instead of saline water and pure water (1).

The real motor of the phenomenon is the difference in hydrostatical pressure resulting from the density difference between the fluids, so the fluids are moved by their weight difference. The system has an initial excess energy because the salt solution, a denser fluid, is above the lighter pure water. It is this potential energy that decreases with each oscillation as downflowing salt water forms a layer on the bottom of the outer vessel and upflowing pure water forms another one on top of the cup. Therefore the oscillation continues until all of the salt water has reached the bottom of the outer vessel, which can take as long as a few days.

Let us have a closer look at what drives the oscillations (see Fig.3).

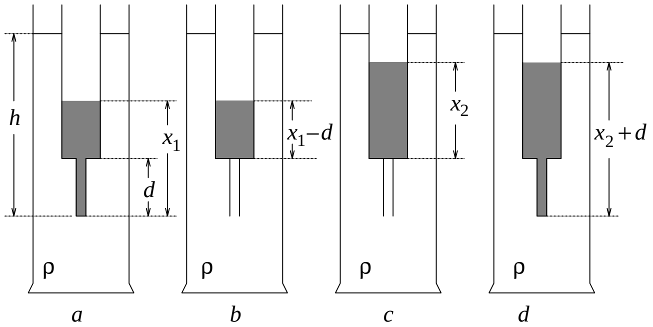


Fig. 3

Starting from the initial state, where the fluids in both the outer and inner vessels are at the same level, salt water begins to pour downwards due to the inequality of hydrostatical pressure. It should be noted that both of the downflowing salt water and the upflowing pure water flow in distinct columns instead of mixing, thus concentration difference at the orifice is kept essentially constant. This downflow continues until the first equilibrium is reached (3a) $x_1 \cdot \varrho_1 = h \cdot \varrho$. At that point the flow ceases, and the pillar of salt water which has formed under the orifice flows down to the bottom of the outer vessel and partly dissolves in the water. Thus a significant concentration difference forms at the bottom of the orifice, and diffusion takes place there. Some ions leave the orifice and mix with the water, therefore average density within the orifice decreases (ions leaving the orifice are moved away from its bottom by Brown's motion and therefore do not affect strongly the density of water there). This upsets the equilibrium (3b): $(x_1 - d) \cdot \varrho_1 < (h - d) \cdot \varrho$ (because $x_1 \cdot \varrho_1 = h \cdot \varrho$ and $d \cdot \varrho_1 > d \cdot \varrho$). Therefore pressure difference at the top of the orifice creates an upward current. (The extent of the decrease of density of the solution within the orifice is irrelevant, since even a change of very small proportions starts a small upflow which carries pure water into the orifice further diluting the solution there, wherefore average density in the orifice diminishes further which makes the upflow stronger etc. which means that we have a self-inducting process). So pure water flows up and into the cup until the level of the solution reaches the height x_2, where the second state of hydrostatical equilibrium is established: $x_2 \cdot \varrho_2 = (h - d) \cdot \varrho$, where ϱ_2 is the average density of the saline water-pure water mixture in the cup (even though the two fluids do not mix, the hydrostatical pressure of the layer of pure water on the surface of the cup adds up to that of the NaCl solution and the easiest way of taking this into account is changing the average density in our calculations). At that point diffusion takes place again, and some ions enter the orifice. This upsets the balance (3c): $(x_2+d)\cdot\varrho_2 > h\cdot\varrho$ (since $x_2\cdot\varrho_2=(h-d)\cdot\varrho_2$ and $d\cdot\varrho_2 > d\cdot\varrho$), and salt water begins to flow down (the current at the beginning is self-inducting again, as before). This downflow stops only when the equilibrium sets in as in (3a). We have reached the initial state again at the end of the oscillation period, with the exception that the layer of pure water on the top of the cup and that of salt water on the bottom of the vessel have grown somewhat thicker, and thus the average density of the salt solution has changed a little.

Experiments with the oscillator

1. — Dependence of period length on height

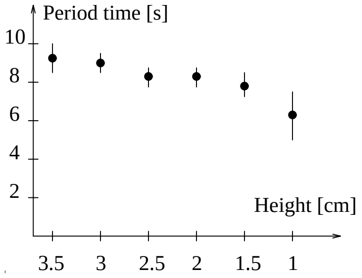


Fig. 4

Height is the distance of the base of the cup and the water level in the outer vessel. We conducted five series of measurements to determine the dependence of period length on height. The cup was held 3.5 cm, 3.0 cm, 2.5 cm, 2.0 cm, 1.5 cm and 1.0 cm under the water level in the outer vessel. Our measurements are shown in Fig. 4.

We found that period length depended linearly on height with a good approximation, and that external effects caused larger fluctuations in period length when height was smaller. The observations are in accordance with our model, because solution level at the states of equilibrium is evidently linearly dependent on height.

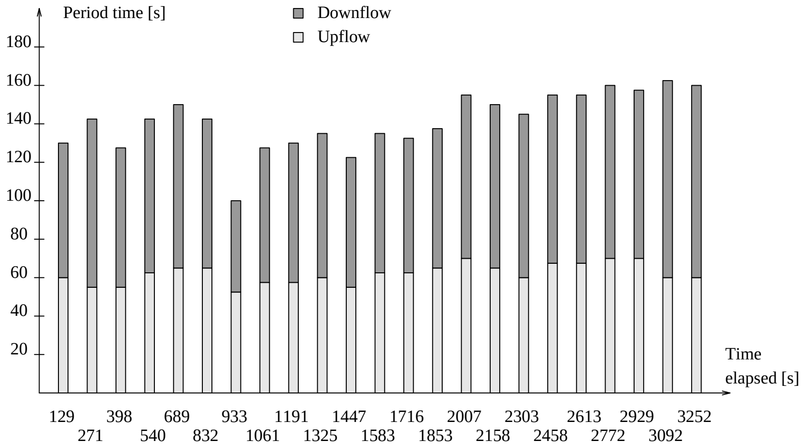


Fig. 5

2. — Dependence of period length on time elapsed

For this experiment we used an oscillator with small orifice diameter (0.5 mm) because that way oscillation periods were longer. Our results are shown in Fig. 5.

We found that the period lengths showed only a marginal increase in the first one hour.

3. — Dependence of period length on concentration

We conducted measurements with three oscillators with NaCl solutions of concentrations 1 mol/l, 1.5 mol/l, and 3 mol/l. Our measurements are shown in Fig. 6.

Dependence of period length on concentration can not sufficiently be determined from our results, and is neither predicted by our model, but we have found one important property of the oscillator, namely that scattering was significantly larger with denser solutions, which shows that they are more sensitive to external effects.

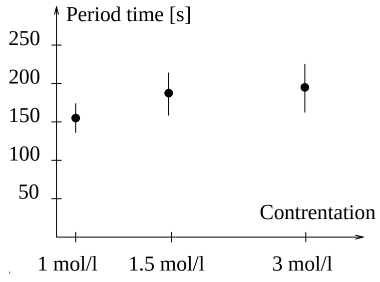


Fig. 6

Comments

1. We have found that period length was affected by orifice diameter to a very large extent: we used an oscillator of larger (1.5 mm) orifice size in experiment 1, whereas in experiments 2 and 3 we used oscillators with small (0.5 mm) orifice sizes. The period lengths for orifices of different sizes differ by two magnitudes: they vary from 7 to 140 seconds. This can be easily explained by our mathematical model: it is the volume of fluids changing places in one oscillation that is a constant, and with larger orifice sizes this exchange takes place more rapidly.

2. Upflow was found to be shorter than downflow. In each oscillation period about 55 % of the time was downflow.

3. There are various other experiments with the saline water oscillator such as coupling oscillators (more cups within the same outer vessel), or investigating oscillators with more orifices.

4. A more accurate mathematical model of the saline oscillator based on nonlinear differential equations is presented in [1].

Reference

[1] Yoshikawa, Oyama, Shoji and Nakata, Am. J. Phys. 59, 137–140.

János Asbóth and **Bálint Laczay,**
Secondary School of the Piarist Fathers of Budapest