

I. megoldás. Az $f(n) = n + 2$ függvény megfelelő, mert értékei pozitív egészek, és tetszőleges n pozitív egészre

$$f(f(n)) + f(n) = ((n + 2) + 2) + (n + 2) = 2n + 6.$$

Megmutatjuk, hogy más megoldás nincs.

Az f függvény biztosan injektív, azaz $n \neq m$ esetén $f(n) \neq f(m)$. Ha ugyanis $f(n) = f(m)$, akkor $f(f(n)) = f(f(m))$ és $2n + 6 = f(f(n)) + f(n) = f(f(m)) + f(m) = 2m + 6$, amiből következik, hogy $n = m$. Az injektivitás következménye, hogy $n \leq h$ esetén $f(n) = 1$, $f(n) = 2$ vagy $f(n) \geq h$.

Most meghatározzuk $f(1)$ értékét. Ha (1)-et felírjuk $n = 1$ -re, megállapíthatunk $f(1)$ -re egy felső korlátot:

$$f(1) = 8 - f(f(1)) \leq 8 - 1 = 7.$$

Az $f(1) = 3$ állítás bizonyításához tehát hat esetet kell kizárni.

I. eset: $f(1) = 1$. Ekkor $f(f(1)) = 1$ és $f(f(1)) + f(1) = 2 \neq 2 \cdot 1 + 6$.

II. eset: $f(1) = 2$. Ekkor

$$f(2) = f(f(1)) = 2 \cdot 1 + 6 - f(1) = 6, f(6) = f(f(2)) = 2 \cdot 2 + 6 - f(2) = 4, f(4) = f(f(6)) = 2 \cdot 6 + 6 - f(6) = 14, f(14) = f(f(4)) = 2 \cdot 4 + 6 - f(4) = 10, f(10) = f(f(14)) = 2 \cdot 14 + 6 - f(14) = 22, f(22) = f(f(10)) = 2 \cdot 10 + 6 - f(10) = 16, f(16) = f(f(22)) = 2 \cdot 22 + 6 - f(22) = 38, f(38) = f(f(16)) = 2 \cdot 16 + 6 - f(16) = 26, f(26) = f(f(38)) = 2 \cdot 38 + 6 - f(38) = 70, f(70) = f(f(26)) = 2 \cdot 26 + 6 - f(26) = 42, f(42) = f(f(70)) = 2 \cdot 70 + 6 - f(70) = 134, f(134) = f(f(42)) = 2 \cdot 42 + 6 - f(42) = 84, f(84) = f(f(134)) = 2 \cdot 134 + 6 - f(134) = 262, f(262) = f(f(84)) = 2 \cdot 84 + 6 - f(84) = 174, f(174) = f(f(262)) = 2 \cdot 262 + 6 - f(262) = 510, f(510) = f(f(174)) = 2 \cdot 174 + 6 - f(174) = 348, f(348) = f(f(510)) = 2 \cdot 510 + 6 - f(510) = 1014, f(1014) = f(f(348)) = 2 \cdot 348 + 6 - f(348) = 702, f(702) = f(f(1014)) = 2 \cdot 1014 + 6 - f(1014) = 2022, f(2022) = f(f(702)) = 2 \cdot 702 + 6 - f(702) = 1404, f(1404) = f(f(2022)) = 2 \cdot 2022 + 6 - f(2022) = 4038, f(4038) = f(f(1404)) = 2 \cdot 1404 + 6 - f(1404) = 2814, f(2814) = f(f(4038)) = 2 \cdot 4038 + 6 - f(4038) = 8070, f(8070) = f(f(2814)) = 2 \cdot 2814 + 6 - f(2814) = 5622, f(5622) = f(f(8070)) = 2 \cdot 8070 + 6 - f(8070) = 16134, f(16134) = f(f(5622)) = 2 \cdot 5622 + 6 - f(5622) = 11244, f(11244) = f(f(16134)) = 2 \cdot 16134 + 6 - f(16134) = 32262, f(32262) = f(f(11244)) = 2 \cdot 11244 + 6 - f(11244) = 22484, f(22484) = f(f(32262)) = 2 \cdot 32262 + 6 - f(32262) = 64510, f(64510) = f(f(22484)) = 2 \cdot 22484 + 6 - f(22484) = 44964, f(44964) = f(f(64510)) = 2 \cdot 64510 + 6 - f(64510) = 128996, f(128996) = f(f(44964)) = 2 \cdot 44964 + 6 - f(44964) = 89914, f(89914) = f(f(128996)) = 2 \cdot 128996 + 6 - f(128996) = 257986, f(257986) = f(f(89914)) = 2 \cdot 89914 + 6 - f(89914) = 179794, f(179794) = f(f(257986)) = 2 \cdot 257986 + 6 - f(257986) = 515954, f(515954) = f(f(179794)) = 2 \cdot 179794 + 6 - f(179794) = 359574, f(359574) = f(f(515954)) = 2 \cdot 515954 + 6 - f(515954) = 1031894, f(1031894) = f(f(359574)) = 2 \cdot 359574 + 6 - f(359574) = 719134, f(719134) = f(f(1031894)) = 2 \cdot 1031894 + 6 - f(1031894) = 2063774, f(2063774) = f(f(719134)) = 2 \cdot 719134 + 6 - f(719134) = 1438254, f(1438254) = f(f(2063774)) = 2 \cdot 2063774 + 6 - f(2063774) = 4127534, f(4127534) = f(f(1438254)) = 2 \cdot 1438254 + 6 - f(1438254) = 2876494, f(2876494) = f(f(4127534)) = 2 \cdot 4127534 + 6 - f(4127534) = 8254994, f(8254994) = f(f(2876494)) = 2 \cdot 2876494 + 6 - f(2876494) = 5752974, f(5752974) = f(f(8254994)) = 2 \cdot 8254994 + 6 - f(8254994) = 16509964, f(16509964) = f(f(5752974)) = 2 \cdot 5752974 + 6 - f(5752974) = 11505934, f(11505934) = f(f(16509964)) = 2 \cdot 16509964 + 6 - f(16509964) = 33019914, f(33019914) = f(f(11505934)) = 2 \cdot 11505934 + 6 - f(11505934) = 23011854, f(23011854) = f(f(33019914)) = 2 \cdot 33019914 + 6 - f(33019914) = 66039814, f(66039814) = f(f(23011854)) = 2 \cdot 23011854 + 6 - f(23011854) = 46023704, f(46023704) = f(f(66039814)) = 2 \cdot 66039814 + 6 - f(66039814) = 132079614, f(132079614) = f(f(46023704)) = 2 \cdot 46023704 + 6 - f(46023704) = 92047404, f(92047404) = f(f(132079614)) = 2 \cdot 132079614 + 6 - f(132079614) = 264159214, f(264159214) = f(f(92047404)) = 2 \cdot 92047404 + 6 - f(92047404) = 184094804, f(184094804) = f(f(264159214)) = 2 \cdot 264159214 + 6 - f(264159214) = 528318414, f(528318414) = f(f(184094804)) = 2 \cdot 184094804 + 6 - f(184094804) = 368189604, f(368189604) = f(f(528318414)) = 2 \cdot 528318414 + 6 - f(528318414) = 1056636814, f(1056636814) = f(f(368189604)) = 2 \cdot 368189604 + 6 - f(368189604) = 736379204, f(736379204) = f(f(1056636814)) = 2 \cdot 1056636814 + 6 - f(1056636814) = 2113273614, f(2113273614) = f(f(736379204)) = 2 \cdot 736379204 + 6 - f(736379204) = 1472758404, f(1472758404) = f(f(2113273614)) = 2 \cdot 2113273614 + 6 - f(2113273614) = 4226547214, f(4226547214) = f(f(1472758404)) = 2 \cdot 1472758404 + 6 - f(1472758404) = 2945516804, f(2945516804) = f(f(4226547214)) = 2 \cdot 4226547214 + 6 - f(4226547214) = 8453094414, f(8453094414) = f(f(2945516804)) = 2 \cdot 2945516804 + 6 - f(2945516804) = 5891033604, f(5891033604) = f(f(8453094414)) = 2 \cdot 8453094414 + 6 - f(8453094414) = 16906168014, f(16906168014) = f(f(5891033604)) = 2 \cdot 5891033604 + 6 - f(5891033604) = 11782067204, f(11782067204) = f(f(16906168014)) = 2 \cdot 16906168014 + 6 - f(16906168014) = 33812336014, f(33812336014) = f(f(11782067204)) = 2 \cdot 11782067204 + 6 - f(11782067204) = 23564134404, f(23564134404) = f(f(33812336014)) = 2 \cdot 33812336014 + 6 - f(33812336014) = 67624672014, f(67624672014) = f(f(23564134404)) = 2 \cdot 23564134404 + 6 - f(23564134404) = 47128268804, f(47128268804) = f(f(67624672014)) = 2 \cdot 67624672014 + 6 - f(67624672014) = 135249344014, f(135249344014) = f(f(47128268804)) = 2 \cdot 47128268804 + 6 - f(47128268804) = 94256587204, f(94256587204) = f(f(135249344014)) = 2 \cdot 135249344014 + 6 - f(135249344014) = 270498688014, f(270498688014) = f(f(94256587204)) = 2 \cdot 94256587204 + 6 - f(94256587204) = 188513374404, f(188513374404) = f(f(270498688014)) = 2 \cdot 270498688014 + 6 - f(270498688014) = 540997376014, f(540997376014) = f(f(188513374404)) = 2 \cdot 188513374404 + 6 - f(188513374404) = 377026748804, f(377026748804) = f(f(540997376014)) = 2 \cdot 540997376014 + 6 - f(540997376014) = 1081994752014, f(1081994752014) = f(f(377026748804)) = 2 \cdot 377026748804 + 6 - f(377026748804) = 754053497604, f(754053497604) = f(f(1081994752014)) = 2 \cdot 1081994752014 + 6 - f(1081994752014) = 2163989504014, f(2163989504014) = f(f(754053497604)) = 2 \cdot 754053497604 + 6 - f(754053497604) = 1508106995204, f(1508106995204) = f(f(2163989504014)) = 2 \cdot 2163989504014 + 6 - f(2163989504014) = 4327979008014, f(4327979008014) = f(f(1508106995204)) = 2 \cdot 1508106995204 + 6 - f(1508106995204) = 3016213990404, f(3016213990404) = f(f(4327979008014)) = 2 \cdot 4327979008014 + 6 - f(4327979008014) = 8655858016014, f(8655858016014) = f(f(3016213990404)) = 2 \cdot 3016213990404 + 6 - f(3016213990404) = 6032427980804, f(6032427980804) = f(f(8655858016014)) = 2 \cdot 8655858016014 + 6 - f(8655858016014) = 17311716032014, f(17311716032014) = f(f(6032427980804)) = 2 \cdot 6032427980804 + 6 - f(6032427980804) = 12064855961604, f(12064855961604) = f(f(17311716032014)) = 2 \cdot 17311716032014 + 6 - f(17311716032014) = 34623432064014, f(34623432064014) = f(f(12064855961604)) = 2 \cdot 12064855961604 + 6 - f(12064855961604) = 24129711923204, f(24129711923204) = f(f(34623432064014)) = 2 \cdot 34623432064014 + 6 - f(34623432064014) = 69246864128014, f(69246864128014) = f(f(24129711923204)) = 2 \cdot 24129711923204 + 6 - f(24129711923204) = 48259423846404, f(48259423846404) = f(f(69246864128014)) = 2 \cdot 69246864128014 + 6 - f(69246864128014) = 138493728256014, f(138493728256014) = f(f(48259423846404)) = 2 \cdot 48259423846404 + 6 - f(48259423846404) = 96518847692804, f(96518847692804) = f(f(138493728256014)) = 2 \cdot 138493728256014 + 6 - f(138493728256014) = 276987456512014, f(276987456512014) = f(f(96518847692804)) = 2 \cdot 96518847692804 + 6 - f(96518847692804) = 193836915385604, f(193836915385604) = f(f(276987456512014)) = 2 \cdot 276987456512014 + 6 - f(276987456512014) = 553974913024014, f(553974913024014) = f(f(193836915385604)) = 2 \cdot 193836915385604 + 6 - f(193836915385604) = 387673830771204, f(387673830771204) = f(f(553974913024014)) = 2 \cdot 553974913024014 + 6 - f(553974913024014) = 1107949826048014, f(1107949826048014) = f(f(387673830771204)) = 2 \cdot 387673830771204 + 6 - f(387673830771204) = 775347661542404, f(775347661542404) = f(f(1107949826048014)) = 2 \cdot 1107949826048014 + 6 - f(1107949826048014) = 2215899652096014, f(2215899652096014) = f(f(775347661542404)) = 2 \cdot 775347661542404 + 6 - f(775347661542404) = 1550695323084804, f(1550695323084804) = f(f(2215899652096014)) = 2 \cdot 2215899652096014 + 6 - f(2215899652096014) = 4431799304192014, f(4431799304192014) = f(f(1550695323084804)) = 2 \cdot 1550695323084804 + 6 - f(1550695323084804) = 3063398646384014, f(3063398646384014) = f(f(4431799304192014)) = 2 \cdot 4431799304192014 + 6 - f(4431799304192014) = 8863598608384014, f(8863598608384014) = f(f(3063398646384014)) = 2 \cdot 3063398646384014 + 6 - f(3063398646384014) = 6126797292768014, f(6126797292768014) = f(f(8863598608384014)) = 2 \cdot 8863598608384014 + 6 - f(8863598608384014) = 17727195216768014, f(17727195216768014) = f(f(6126797292768014)) = 2 \cdot 6126797292768014 + 6 - f(6126797292768014) = 12253594585536014, f(12253594585536014) = f(f(17727195216768014)) = 2 \cdot 17727195216768014 + 6 - f(17727195216768014) = 35454389131536014, f(35454389131536014) = f(f(12253594585536014)) = 2 \cdot 12253594585536014 + 6 - f(12253594585536014) = 24507179171072014, f(24507179171072014) = f(f(35454389131536014)) = 2 \cdot 35454389131536014 + 6 - f(35454389131536014) = 70908778263072014, f(70908778263072014) = f(f(24507179171072014)) = 2 \cdot 24507179171072014 + 6 - f(24507179171072014) = 49014358342144014, f(49014358342144014) = f(f(70908778263072014)) = 2 \cdot 70908778263072014 + 6 - f(70908778263072014) = 141817556526288014, f(141817556526288014) = f(f(49014358342144014)) = 2 \cdot 49014358342144014 + 6 - f(49014358342144014) = 98028716684288014, f(98028716684288014) = f(f(141817556526288014)) = 2 \cdot 141817556526288014 + 6 - f(141817556526288014) = 283635113052576014, f(283635113052576014) = f(f(98028716684288014)) = 2 \cdot 98028716684288014 + 6 - f(98028716684288014) = 196057433368576014, f(196057433368576014) = f(f(283635113052576014)) = 2 \cdot 283635113052576014 + 6 - f(283635113052576014) = 567270226105152014, f(567270226105152014) = f(f(196057433368576014)) = 2 \cdot 196057433368576014 + 6 - f(196057433368576014) = 392114866737152014, f(392114866737152014) = f(f(567270226105152014)) = 2 \cdot 567270226105152014 + 6 - f(567270226105152014) = 1134540452274304014, f(1134540452274304014) = f(f(392114866737152014)) = 2 \cdot 392114866737152014 + 6 - f(392114866737152014) = 784229733474304014, f(784229733474304014) = f(f(1134540452274304014)) = 2 \cdot 1134540452274304014 + 6 - f(1134540452274304014) = 2269080904548608014, f(2269080904548608014) = f(f(784229733474304014)) = 2 \cdot 784229733474304014 + 6 - f(784229733474304014) = 1568459469097608014, f(1568459469097608014) = f(f(2269080904548608014)) = 2 \cdot 2269080904548608014 + 6 - f(2269080904548608014) = 4538161819197216014, f(4538161819197216014) = f(f(1568459469097608014)) = 2 \cdot 1568459469097608014 + 6 - f(1568459469097608014) = 3136913738394432014, f(3136913738394432014) = f(f(4538161819197216014)) = 2 \cdot 4538161819197216014 + 6 - f(4538161819197216014) = 9076323638392864014, f(9076323638392864014) = f(f(3136913738394432014)) = 2 \cdot 3136913738394432014 + 6 - f(3136913738394432014) = 6273847476788864014, f(6273847476788864014) = f(f(9076323638392864014)) = 2 \cdot 9076323638392864014 + 6 - f(9076323638392864014) = 18152647276777728014, f(18152647276777728014) = f(f(6273847476788864014)) = 2 \cdot 6273847476788864014 + 6 - f(6273847476788864014) = 12547694553575728014, f(12547694553575728014) = f(f(18152647276777728014)) = 2 \cdot 18152647276777728014 + 6 - f(18152647276777728014) = 36305394553551456014, f(36305394553551456014) = f(f(12547694553575728014)) = 2 \cdot 12547694553575728014 + 6 - 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Ha most $m \rightarrow \infty$, akkor az alsó és a felső becslés is 0-hoz tart, tehát $b_1 = 0$.

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dolgozata alapján

Megjegyzések. 1. A második megoldás valójában az elsőnek egy általánosított változata. Az első megoldás esetszét-választásait egyszerre számoltuk ki. Például ha $f(1) = x$, akkor a függvényegyenlet alapján megállapíthatjuk, hogy $f(x) = 8 - x$, $f(8 - x) = 3x - 2$, $f(3x - 2) = 24 - 5x$, $f(24 - 5x) = 11x - 22$, $f(11x - 22) = 76 - 21x$ stb. Ahhoz, hogy az utolsó két szám pozitív egész legyen, szükséges, hogy $11x - 22 \geq 1$ és $76 - 21x \geq 1$, azaz $2\frac{1}{11} \leq x \leq 3\frac{4}{7}$ legyen. Ebben az intervallumban az egyetlen egész szám a 3.

2. A második megoldás akkor is működik, ha az f függvény pozitív *valós* számok halmazát képezi önmagára. Ilyenkor is $f(x) = x + 2$ az egyetlen megoldás.