Second round

1. The hyperbolas with equations $xy = 1$ and $xy = -1$ are drawn on the same set of coordinate axes. Prove that if the intersection of a circle of radius R centred at the origin with the hyperbolas form the vertex set of a regular polygon then the area of the polygon is equal to R^4.

2. Find the real numbers x, y, z, t satisfying the following simultaneous equation and inequality:

$$x + y + z = \frac{3}{2},$$

$$\sqrt{4x-1} + \sqrt{4y-1} + \sqrt{4z-1} \geq 2 + 3^{\sqrt{t-2}}.$$

3. Let $f(x)$ denote a real valued function defined on the set of real numbers different from 0 and 1. Determine the function $f(x)$, such that the equation

$$f(x) + kx^2 f\left(\frac{1}{x}\right) = \frac{x}{x+1}$$

is satisfied by every x in its domain, where k is a constant with $0 < k^2 \neq 1$. For what elements x in the domain is $f(x) = 0$?

4. Let O_A denote the point where the bisector of the interior angle at vertex A of a triangle ABC first intersects the inscribed circle. Obtain the points O_B and O_C similarly, on the interior angle bisectors from B and C. Let k_A be the circle about O_A, tangent to AB and CA. Similarly, let k_B denote the circle about O_B touching BC and AB and let k_C denote the circle about O_C touching CA and BC.

Prove that the three lines different from the sides of the triangle that touch two of the circles k_A, k_B, k_C externally are concurrent.

Third (final) round

1. The excircle touching the side AB of a triangle ABC touches AB at P and the extension of AC at Q. The excircle drawn to side BC touches the extension of AC at U and the extension of AB at X.

Prove that the intersection of the lines PQ and UX is equidistant from the lines AB and BC.

2. Is there an n-sided polygon in which the number of acute angles is

$$n^2 - 30n + 236?$$

3. Let n be a fixed integer greater than 1. Find real numbers $x_1, x_2, \dots, x_n$ such that

$$x_1 + x_2 + \cdots + x_n = 2(n-1),$$

$$(x_1 - 1)^2 + (x_2 - 1)^2 + \cdots + (x_n - 1)^2 = n,$$

and x_n is as large as possible.

Schools with advanced mathematics programme

First round

1. Let $a = 1 + \sqrt{5}$. Evaluate $(4 - a) \cdot \sqrt{2 + a} \cdot \sqrt[3]{a} \cdot \sqrt[6]{3a + 4}$.

2. $ABCD$ is a trapezium with parallel sides AB and CD. Let E and F be interior points on the sides AD and BC, respectively. Show that if the lines AF and EC are parallel then so are the lines EB and DF.

3. Given that the representation of a certain power of two consists of identical digits in a particular number system, prove that the representation has at most two digits.

4. Is there a non-constant polynomial with integer coefficients that assigns a value of the form $k!$ to every positive integer (where k is a positive integer)?

5. Determine the expected value of the second largest number drawn in lottery in which 5 numbers are drawn at random from 1 to 90 inclusive. [The expected value: Consider the second largest number in every possible set of numbers drawn, and take their arithmetic mean. (A certain number is counted as many times as it occurs as second largest number in the different draws.)]

Second (final) round

1. Each x_i out of the numbers $x_1, x_2, \ldots, x_n$ ($n \geq 1$) equals the sum of the squares of the remaining numbers x_j. Determine all possible such sequences of n numbers.

2. Select an interior point on each side of a parallelogram. Prove that the perimeter of the quadrilateral determined by the four points is at least twice as long as the shorter diagonal of the parallelogram.

3. Find a subset H of the set of positive integers that has the following two properties:

(i) every sufficiently large positive integer can be expressed as a sum of *at most* 100 (not necessarily different) elements of H;

(ii) 2002 is the smallest number k, such that every sufficiently large positive integer can be expressed as a sum of *exactly* k (not necessarily different) elements of H.