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3. *Determine all pairs (m, n) of integers $m, n \geq 3$, such that there exist infinitely many positive integers a for which*

$$\frac{a^m + a - 1}{a^n + a^2 - 1}$$

is an integer.

Solution by B. A. Rácz. Let $\mathrm{p}(x) = x^m + x - 1$ and $\mathrm{q}(x) = x^n + x^2 - 1$. First we shall prove that if the condition of the problem is satisfied by a pair (m, n) then the denominator as a polynomial is a divisor of the numerator, that is $\mathrm{q}(x) \mid \mathrm{p}(x)$.

Let us divide the polynomial $\mathrm{p}(x)$ by $\mathrm{q}(x)$, that is, let

$$\mathrm{p}(x) = \mathrm{h}(x) \cdot \mathrm{q}(x) + \mathrm{r}(x),$$

where either $r(x)$ is zero or the degree of the remainder is smaller than that of the divisor: $\deg \mathrm{r} < \deg \mathrm{q}$. Since the leading coefficient of the divisor is 1, the quotient and remainder both have integer coefficients.

According to the condition, it is true for infinitely many integers a that

$$\frac{\mathrm{p}(a)}{\mathrm{q}(a)} = \mathrm{h}(a) + \frac{\mathrm{r}(a)}{\mathrm{q}(a)}$$

is an integer. It follows from the relation of the degrees that $\dfrac{\mathrm{r}(a)}{\mathrm{q}(a)} \to 0$ as $|a| \to \infty$, and thus the value of this ratio must be 0 for infinitely many integers a. This is also true for the numerator, and if the value of the polynomial is 0 at infinitely many points then it must be identically zero.

Thus the polynomial $\mathrm{r}(x)$ is identically zero, and $\mathrm{q}(x)$ is a divisor of the polynomial $\mathrm{p}(x)$.

Since the divisibility cannot hold if $m < n$, we can assume that $m \geq n$. Then the polynomial $\mathrm{q}(x)$ also divides the polynomial

$$(x+1) \cdot \mathrm{p}(x) - \mathrm{q}(x) = x^n(x^{m-n+1} + x^{m-n} - 1).$$

Since x^n and $x^n + x^2 - 1$ are coprime and the second factor is also a polynomial according to the assumption, it follows that

$$x^n + x^2 - 1 \mid x^{m-n+1} + x^{m-n} - 1.$$

Let $k = m - n$. Then $k \geq 0$, $\mathrm{q}(x) = x^n + x^2 - 1 \mid x^{k+1} + x^k - 1$ and it is clear that $k + 1 \geq n$.

Since $\mathrm{q}(x)$ is a continuous function and $\mathrm{q}(0) < 0 < \mathrm{q}(1)$, there exists a number $0 < \alpha < 1$, such that $\mathrm{q}(\alpha) = 0$, that is $\alpha^n + \alpha^2 = 1$. Hence it follows from the divisibility $\mathrm{q}(x) \mid x^{k+1} + x^k - 1$ that $\alpha^{k+1} + \alpha^k = 1$, and thus

$$(*) \qquad \alpha^n + \alpha^2 = \alpha^{k+1} + \alpha^k = 1.$$

If $k = 1$, the second equality cannot be true for any real α. If $k \geq 2$ then $n \geq 3$ and thus with the condition $k + 1 \geq n$ above, we have

$$k \geq n - 1 \geq 2.$$

Since $0 < \alpha < 1$, it follows that $\alpha^n \geq \alpha^{k+1}$ and $\alpha^2 \geq \alpha^k$. With $(*)$, this can happen only if $\alpha^n = \alpha^{k+1}$ and $\alpha^2 = \alpha^k$, that is for $n = k + 1$ and $2 = k$, which makes $m = 5$ and $n = 3$.

For this number pair, on the other hand, $a^5 + a - 1 = (a^3 + a^2 - 1)(a^2 - a + 1)$. If a is a positive integer then $a^3 + a^2 - 1 \geq 1 + 1 - 1 > 0$, and thus

$$\frac{a^5 + a - 1}{a^3 + a^2 - 1} = a^2 - a + 1,$$

which is an integer for every positive integer a.

There is one single pair of numbers satisfying the conditions of the problem, namely the pair $(5, 3)$.