

**I. megoldás.** Az  $f(n) = n + 2$  függvény megfelelő, mert értékei pozitív egészek, és tetszőleges  $n$  pozitív egészre

$$f(f(n)) + f(n) = ((n + 2) + 2) + (n + 2) = 2n + 6.$$

Megmutatjuk, hogy más megoldás nincs.

Az  $f$  függvény biztosan injektív, azaz  $n \neq m$  esetén  $f(n) \neq f(m)$ . Ha ugyanis  $f(n) = f(m)$ , akkor  $f(f(n)) = f(f(m))$  és  $2n + 6 = f(f(n)) + f(n) = f(f(m)) + f(m) = 2m + 6$ , amiből következik, hogy  $n = m$ . Az injektivitás következménye, hogy  $n \leq h$  esetén  $f(n) = 1$ ,  $f(n) = 2$  vagy  $f(n) \geq h$ .

Most meghatározzuk  $f(1)$  értékét. Ha (1)-et felírjuk  $n = 1$ -re, megállapíthatunk  $f(1)$ -re egy felső korlátot:

$$f(1) = 8 - f(f(1)) \leq 8 - 1 = 7.$$

Az  $f(1) = 3$  állítás bizonyításához tehát hat esetet kell kizárni.

I. eset:  $f(1) = 1$ . Ekkor  $f(f(1)) = 1$  és  $f(f(1)) + f(1) = 2 \neq 2 \cdot 1 + 6$ .

II. eset:  $f(1) = 2$ . Ekkor

$$f(2) = f(f(1)) = 2 \cdot 1 + 6 - f(1) = 6, f(6) = f(f(2)) = 2 \cdot 2 + 6 - f(2) = 4, f(4) = f(f(6)) = 2 \cdot 6 + 6 - f(6) = 14, f(14) = f(f(4)) = 2 \cdot 4 + 6 - f(4) = 10, f(10) = f(f(14)) = 2 \cdot 14 + 6 - f(14) = 22, f(22) = f(f(10)) = 2 \cdot 10 + 6 - f(10) = 16, f(16) = f(f(22)) = 2 \cdot 22 + 6 - f(22) = 38, f(38) = f(f(16)) = 2 \cdot 16 + 6 - f(16) = 26, f(26) = f(f(38)) = 2 \cdot 38 + 6 - f(38) = 70, f(70) = f(f(26)) = 2 \cdot 26 + 6 - f(26) = 46, f(46) = f(f(70)) = 2 \cdot 70 + 6 - f(70) = 134, f(134) = f(f(46)) = 2 \cdot 46 + 6 - f(46) = 88, f(88) = f(f(134)) = 2 \cdot 134 + 6 - f(134) = 262, f(262) = f(f(88)) = 2 \cdot 88 + 6 - f(88) = 170, f(170) = f(f(262)) = 2 \cdot 262 + 6 - f(262) = 518, f(518) = f(f(170)) = 2 \cdot 170 + 6 - f(170) = 340, f(340) = f(f(518)) = 2 \cdot 518 + 6 - f(518) = 1030, f(1030) = f(f(340)) = 2 \cdot 340 + 6 - f(340) = 680, f(680) = f(f(1030)) = 2 \cdot 1030 + 6 - f(1030) = 2054, f(2054) = f(f(680)) = 2 \cdot 680 + 6 - f(680) = 1360, f(1360) = f(f(2054)) = 2 \cdot 2054 + 6 - f(2054) = 4098, f(4098) = f(f(1360)) = 2 \cdot 1360 + 6 - f(1360) = 2716, f(2716) = f(f(4098)) = 2 \cdot 4098 + 6 - f(4098) = 8190, f(8190) = f(f(2716)) = 2 \cdot 2716 + 6 - f(2716) = 5428, f(5428) = f(f(8190)) = 2 \cdot 8190 + 6 - f(8190) = 16374, f(16374) = f(f(5428)) = 2 \cdot 5428 + 6 - f(5428) = 10858, f(10858) = f(f(16374)) = 2 \cdot 16374 + 6 - f(16374) = 32742, f(32742) = f(f(10858)) = 2 \cdot 10858 + 6 - f(10858) = 21712, f(21712) = f(f(32742)) = 2 \cdot 32742 + 6 - f(32742) = 65478, f(65478) = f(f(21712)) = 2 \cdot 21712 + 6 - f(21712) = 43420, f(43420) = f(f(65478)) = 2 \cdot 65478 + 6 - f(65478) = 130950, f(130950) = f(f(43420)) = 2 \cdot 43420 + 6 - f(43420) = 86836, f(86836) = f(f(130950)) = 2 \cdot 130950 + 6 - f(130950) = 261894, f(261894) = f(f(86836)) = 2 \cdot 86836 + 6 - f(86836) = 173668, f(173668) = f(f(261894)) = 2 \cdot 261894 + 6 - f(261894) = 523782, f(523782) = f(f(173668)) = 2 \cdot 173668 + 6 - f(173668) = 347336, f(347336) = f(f(523782)) = 2 \cdot 523782 + 6 - f(523782) = 1047560, f(1047560) = f(f(347336)) = 2 \cdot 347336 + 6 - f(347336) = 694668, f(694668) = f(f(1047560)) = 2 \cdot 1047560 + 6 - f(1047560) = 2095114, f(2095114) = f(f(694668)) = 2 \cdot 694668 + 6 - f(694668) = 1389336, f(1389336) = f(f(2095114)) = 2 \cdot 2095114 + 6 - f(2095114) = 4190222, f(4190222) = f(f(1389336)) = 2 \cdot 1389336 + 6 - f(1389336) = 2778672, f(2778672) = f(f(4190222)) = 2 \cdot 4190222 + 6 - f(4190222) = 8380440, f(8380440) = f(f(2778672)) = 2 \cdot 2778672 + 6 - f(2778672) = 5557344, f(5557344) = f(f(8380440)) = 2 \cdot 8380440 + 6 - f(8380440) = 16760880, f(16760880) = f(f(5557344)) = 2 \cdot 5557344 + 6 - f(5557344) = 11114682, f(11114682) = f(f(16760880)) = 2 \cdot 16760880 + 6 - f(16760880) = 33521760, f(33521760) = f(f(11114682)) = 2 \cdot 11114682 + 6 - f(11114682) = 22229364, f(22229364) = f(f(33521760)) = 2 \cdot 33521760 + 6 - f(33521760) = 67043514, f(67043514) = f(f(22229364)) = 2 \cdot 22229364 + 6 - f(22229364) = 44458722, f(44458722) = f(f(67043514)) = 2 \cdot 67043514 + 6 - f(67043514) = 134087022, f(134087022) = f(f(44458722)) = 2 \cdot 44458722 + 6 - f(44458722) = 88917444, f(88917444) = f(f(134087022)) = 2 \cdot 134087022 + 6 - f(134087022) = 268174040, f(268174040) = f(f(88917444)) = 2 \cdot 88917444 + 6 - f(88917444) = 177834882, f(177834882) = f(f(268174040)) = 2 \cdot 268174040 + 6 - f(268174040) = 536348074, f(536348074) = f(f(177834882)) = 2 \cdot 177834882 + 6 - f(177834882) = 355669760, f(355669760) = f(f(536348074)) = 2 \cdot 536348074 + 6 - f(536348074) = 1072696142, f(1072696142) = f(f(355669760)) = 2 \cdot 355669760 + 6 - f(355669760) = 711339516, f(711339516) = f(f(1072696142)) = 2 \cdot 1072696142 + 6 - f(1072696142) = 2145392280, f(2145392280) = f(f(711339516)) = 2 \cdot 711339516 + 6 - f(711339516) = 1422679032, f(1422679032) = f(f(2145392280)) = 2 \cdot 2145392280 + 6 - f(2145392280) = 4290784560, f(4290784560) = f(f(1422679032)) = 2 \cdot 1422679032 + 6 - f(1422679032) = 2845358064, f(2845358064) = f(f(4290784560)) = 2 \cdot 4290784560 + 6 - f(4290784560) = 8581569114, f(8581569114) = f(f(2845358064)) = 2 \cdot 2845358064 + 6 - f(2845358064) = 5690716122, f(5690716122) = f(f(8581569114)) = 2 \cdot 8581569114 + 6 - f(8581569114) = 17163138222, f(17163138222) = f(f(5690716122)) = 2 \cdot 5690716122 + 6 - f(5690716122) = 11381432240, f(11381432240) = f(f(17163138222)) = 2 \cdot 17163138222 + 6 - f(17163138222) = 34326276440, f(34326276440) = f(f(11381432240)) = 2 \cdot 11381432240 + 6 - f(11381432240) = 22762864474, f(22762864474) = f(f(34326276440)) = 2 \cdot 34326276440 + 6 - f(34326276440) = 68652552874, f(68652552874) = f(f(22762864474)) = 2 \cdot 22762864474 + 6 - f(22762864474) = 45525728942, f(45525728942) = f(f(68652552874)) = 2 \cdot 68652552874 + 6 - f(68652552874) = 137305105742, f(137305105742) = f(f(45525728942)) = 2 \cdot 45525728942 + 6 - f(45525728942) = 91051457874, f(91051457874) = f(f(137305105742)) = 2 \cdot 137305105742 + 6 - f(137305105742) = 274610211480, f(274610211480) = f(f(91051457874)) = 2 \cdot 91051457874 + 6 - f(91051457874) = 182102915742, f(182102915742) = f(f(274610211480)) = 2 \cdot 274610211480 + 6 - f(274610211480) = 549220422960, f(549220422960) = f(f(182102915742)) = 2 \cdot 182102915742 + 6 - f(182102915742) = 364205831480, f(364205831480) = f(f(549220422960)) = 2 \cdot 549220422960 + 6 - f(549220422960) = 1098440845914, f(1098440845914) = f(f(364205831480)) = 2 \cdot 364205831480 + 6 - f(364205831480) = 728411662960, f(728411662960) = f(f(1098440845914)) = 2 \cdot 1098440845914 + 6 - f(1098440845914) = 2196881691822, f(2196881691822) = f(f(728411662960)) = 2 \cdot 728411662960 + 6 - f(728411662960) = 1456823325814, f(1456823325814) = f(f(2196881691822)) = 2 \cdot 2196881691822 + 6 - f(2196881691822) = 4393763383640, f(4393763383640) = f(f(1456823325814)) = 2 \cdot 1456823325814 + 6 - f(1456823325814) = 2913646651622, f(2913646651622) = f(f(4393763383640)) = 2 \cdot 4393763383640 + 6 - f(4393763383640) = 8787526767274, f(8787526767274) = f(f(2913646651622)) = 2 \cdot 2913646651622 + 6 - f(2913646651622) = 5827293303240, f(5827293303240) = f(f(8787526767274)) = 2 \cdot 8787526767274 + 6 - f(8787526767274) = 17575053534540, f(17575053534540) = f(f(5827293303240)) = 2 \cdot 5827293303240 + 6 - f(5827293303240) = 11654586606474, f(11654586606474) = f(f(17575053534540)) = 2 \cdot 17575053534540 + 6 - f(17575053534540) = 35150107069074, f(35150107069074) = f(f(11654586606474)) = 2 \cdot 11654586606474 + 6 - f(11654586606474) = 23309173212942, f(23309173212942) = f(f(35150107069074)) = 2 \cdot 35150107069074 + 6 - f(35150107069074) = 70300214138142, f(70300214138142) = f(f(23309173212942)) = 2 \cdot 23309173212942 + 6 - f(23309173212942) = 46618346425874, f(46618346425874) = f(f(70300214138142)) = 2 \cdot 70300214138142 + 6 - f(70300214138142) = 140600428276274, f(140600428276274) = f(f(46618346425874)) = 2 \cdot 46618346425874 + 6 - f(46618346425874) = 93236692851742, f(93236692851742) = f(f(140600428276274)) = 2 \cdot 140600428276274 + 6 - f(140600428276274) = 281200856552540, f(281200856552540) = f(f(93236692851742)) = 2 \cdot 93236692851742 + 6 - f(93236692851742) = 186473385703474, f(186473385703474) = f(f(281200856552540)) = 2 \cdot 281200856552540 + 6 - f(281200856552540) = 562401713105074, f(562401713105074) = f(f(186473385703474)) = 2 \cdot 186473385703474 + 6 - f(186473385703474) = 372946771406942, f(372946771406942) = f(f(562401713105074)) = 2 \cdot 562401713105074 + 6 - f(562401713105074) = 1124803426210142, f(1124803426210142) = f(f(372946771406942)) = 2 \cdot 372946771406942 + 6 - f(372946771406942) = 745893542813874, f(745893542813874) = f(f(1124803426210142)) = 2 \cdot 1124803426210142 + 6 - f(1124803426210142) = 2249606852420274, f(2249606852420274) = f(f(745893542813874)) = 2 \cdot 745893542813874 + 6 - f(745893542813874) = 1491787085627742, f(1491787085627742) = f(f(2249606852420274)) = 2 \cdot 2249606852420274 + 6 - f(2249606852420274) = 4499213704841540, f(4499213704841540) = f(f(1491787085627742)) = 2 \cdot 1491787085627742 + 6 - f(1491787085627742) = 2983526141273274, f(2983526141273274) = f(f(4499213704841540)) = 2 \cdot 4499213704841540 + 6 - f(4499213704841540) = 8998427409683074, f(8998427409683074) = f(f(2983526141273274)) = 2 \cdot 2983526141273274 + 6 - f(2983526141273274) = 5967052282546542, f(5967052282546542) = f(f(8998427409683074)) = 2 \cdot 8998427409683074 + 6 - f(8998427409683074) = 17996854819366142, f(17996854819366142) = f(f(5967052282546542)) = 2 \cdot 5967052282546542 + 6 - f(5967052282546542) = 11934104564893074, f(11934104564893074) = f(f(17996854819366142)) = 2 \cdot 17996854819366142 + 6 - f(17996854819366142) = 35993709638732274, f(35993709638732274) = f(f(11934104564893074)) = 2 \cdot 11934104564893074 + 6 - f(11934104564893074) = 23868209129784542, f(23868209129784542) = f(f(35993709638732274)) = 2 \cdot 35993709638732274 + 6 - f(35993709638732274) = 71987419277464542, f(71987419277464542) = f(f(23868209129784542)) = 2 \cdot 23868209129784542 + 6 - f(23868209129784542) = 47736418259569074, f(47736418259569074) = f(f(71987419277464542)) = 2 \cdot 71987419277464542 + 6 - f(71987419277464542) = 143974838554939074, f(143974838554939074) = f(f(47736418259569074)) = 2 \cdot 47736418259569074 + 6 - f(47736418259569074) = 95472836519138142, f(95472836519138142) = f(f(143974838554939074)) = 2 \cdot 143974838554939074 + 6 - f(143974838554939074) = 287949677109878142, f(287949677109878142) = f(f(95472836519138142)) = 2 \cdot 95472836519138142 + 6 - f(95472836519138142) = 190945654238356274, f(190945654238356274) = f(f(287949677109878142)) = 2 \cdot 287949677109878142 + 6 - f(287949677109878142) = 575899354219756274, f(575899354219756274) = f(f(190945654238356274)) = 2 \cdot 190945654238356274 + 6 - f(190945654238356274) = 381891308476712542, f(381891308476712542) = f(f(575899354219756274)) = 2 \cdot 575899354219756274 + 6 - f(575899354219756274) = 1151798708439512542, f(1151798708439512542) = f(f(381891308476712542)) = 2 \cdot 381891308476712542 + 6 - f(381891308476712542) = 763782616953425074, f(763782616953425074) = f(f(1151798708439512542)) = 2 \cdot 1151798708439512542 + 6 - f(1151798708439512542) = 2303597416879025074, f(2303597416879025074) = f(f(763782616953425074)) = 2 \cdot 763782616953425074 + 6 - f(763782616953425074) = 1527565233806850142, f(1527565233806850142) = f(f(2303597416879025074)) = 2 \cdot 2303597416879025074 + 6 - f(2303597416879025074) = 4607194833757650142, f(4607194833757650142) = f(f(1527565233806850142)) = 2 \cdot 1527565233806850142 + 6 - f(1527565233806850142) = 3055188667513700274, f(3055188667513700274) = f(f(4607194833757650142)) = 2 \cdot 4607194833757650142 + 6 - f(4607194833757650142) = 9214389667517400274, f(9214389667517400274) = f(f(3055188667513700274)) = 2 \cdot 3055188667513700274 + 6 - f(3055188667513700274) = 6109379335035400542, f(6109379335035400542) = f(f(9214389667517400274)) = 2 \cdot 9214389667517400274 + 6 - f(9214389667517400274) = 18428779335035400542, f(18428779335035400542) = f(f(6109379335035400542)) = 2 \cdot 6109379335035400542 + 6 - f(6109379335035400542) = 36857558670070801074, f(36857558670070801074) = f(f(18428779335035400542)) = 2 \cdot 18428779335035400542 + 6 - f(18428779335035400542) = 73715117340141602142, f(73715117340141602142) = f(f(36857558670070801074)) = 2 \cdot 36857558670070801074 + 6 - f(36857558670070801074) = 147430234680283204274, f(147430234680283204274) = f(f(73715117340141602142)) = 2 \cdot 73715117340141602142 + 6 - f(73715117340141602142) = 294860469360566408542, f(294860469360566408542) = f(f(147430234680283204274)) = 2 \cdot 147430234680283204274 + 6 - f(147430234680283204274) = 589720938721132817074, f(589720938721132817074) = f(f(294860469360566408542)) = 2 \cdot 294860469360566408542 + 6 - f(294860469360566408542) = 1179441877442265634142, f(1179441877442265634142) = f(f(589720938721132817074)) = 2 \cdot 589720938721132817074 + 6 - f(589720938721132817074) = 2358883754884531268274, f(2358883754884531268274) = f(f(1179441877442265634142)) = 2 \cdot 1179441877442265634142 + 6 - f(1179441877442265634142) = 4717767513769062536542, f(4717767513769062536542) = f(f(2358883754884531268274)) = 2 \cdot 2358883754884531268274 + 6 - f(2358883754884531268274) = 9435535027538125173074, f(9435535027538125173074) = f(f(4717767513769062536542)) = 2 \cdot 4717767513769062536542 + 6 - f(4717767513769062536542) = 18871070055076250347142, f(18871070055076250347142) = f(f(9435535027538125173074)) = 2 \cdot 9435535027538125173074 + 6 - f(9435535027538125173074) = 37742140110152500694274, f(37742140110152500694274) = f(f(18871070055076250347142)) = 2 \cdot 18871070055076250347142 + 6 - f(18871070055076250347142) = 75484280220305001388542, f(75484280220305001388542) = f(f(37742140110152500694274)) = 2 \cdot 37742140110152500694274 + 6 - f(37742140110152500694274) = 150968560440610002777142, f(150968560440610002777142) = f(f(75484280220305001388542)) = 2 \cdot 75484280220305001388542 + 6 - f(75484280220305001388542) = 301937120881220005554274, f(301937120881220005554274) = f(f(150968560440610002777142)) = 2 \cdot 150968560440610002777142 + 6 - f(150968560440610002777142) = 603874241762440005554274, f(603874241762440005554274) = f(f(301937120881220005554274)) = 2 \cdot 301937120881220005554274 + 6 - f(301937120881220005554274) = 1207748483524880005554274, f(1207748483524880005554274) = f(f(603874241762440005554274)) = 2 \cdot 603874241762440005554274 + 6 - f(603874241762440005554$$

Ha most  $m \rightarrow \infty$ , akkor az alsó és a felső becslés is 0-hoz tart, tehát  $b_1 = 0$ .

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dolgozata alapján

*Megjegyzések.* 1. A második megoldás valójában az elsőnek egy általánosított változata. Az első megoldás esetszét-választásait egyszerre számoltuk ki. Például ha  $f(1) = x$ , akkor a függvényegyenlet alapján megállapíthatjuk, hogy  $f(x) = 8 - x$ ,  $f(8 - x) = 3x - 2$ ,  $f(3x - 2) = 24 - 5x$ ,  $f(24 - 5x) = 11x - 22$ ,  $f(11x - 22) = 76 - 21x$  stb. Ahhoz, hogy az utolsó két szám pozitív egész legyen, szükséges, hogy  $11x - 22 \geq 1$  és  $76 - 21x \geq 1$ , azaz  $2\frac{1}{11} \leq x \leq 3\frac{4}{7}$  legyen. Ebben az intervallumban az egyetlen egész szám a 3.

2. A második megoldás akkor is működik, ha az  $f$  függvény pozitív *valós* számok halmazát képezi önmagára. Ilyenkor is  $f(x) = x + 2$  az egyetlen megoldás.