In the previous issue of KöMaL, we published the questions on the pools of the Autumn Conference, 2003. The answers are

$$\mathrm{X},\mathrm{X},1,\qquad 1,\mathrm{X},1,\qquad \mathrm{X},1,2,\qquad 1,1,2,\qquad 1,\mathrm{X}.$$

No one scored 13+1. Those who scored 12 were awarded a book: *Zs. Jankó* (10th grade, Szeged); *E. Csóka* (university student); *M. Horváth, T. Hubai* and *D. Paulin* (12th grade, Budapest).

1. (This question was about the meaning of a rarely used Hungarian word, a single word for „two and a half". Since an exact translation is impossible, here is an analogous question instead:)

What number does „three score" mean? (1) $\frac{1}{3}$; (2) 30; (X) 60.

Solution. Score means a set (or group) of 20. Three score is 60.

2. *The density of a homogeneous cube is* $500\ \mathrm{kg/m^3}$. *If it is placed on the surface of water, it will float* (1) *with two faces in horizontal position;* (2) *with two edges exactly on the surface but no face horizontal*; (X) *in some other position.*

Solution. The position of stable floating is determined by the *minimum* of the gravitational potential energy of the system (of water and object.) Since exactly half of the volume of the cube is submerged, its centre of mass will be at the level of the water surface. Thus the potential energy of the cube is independent of the orientation of its faces relative to the water. It is *always 0* if the surface is chosen to be the 0 level of energy.

The potential energy of the water (if there were water in the volume filled by the submerged part of the cube) would also be independent of the position of the cube. Thus the potential energy of the *whole system* equals (-1) times the energy of the displaced water, that is proportional to the distance d of the centre of mass of the displaced water from the surface. Therefore the floating of the cube will be stable if the centre of mass of the submerged part is in the highest possible position.

When two faces are horizontal, the submerged part is a square prism and $d=d_1=\frac{1}{4}a=0.25\,a$ (where a is the length of the edge of the cube). A lower potential energy is obtained if four edges of the cube are horizontal, two lying on the surface of the water. Then the submerged part is a triangular prism, and

$$d=d_2=\frac{\sqrt{2}}{6}a\approx 0.2357\,a<d_1.$$

However, potential energy is even lower if one diagonal of the cube is vertical and the intersection with the plane of the water surface is a regular hexagon. Then

$$d=d_3=\frac{13\sqrt{3}}{96}a\approx 0.2345\,a<d_2.$$

It can be shown that the positions corresponding to d_1 and d_2 are unstable for small displacements, while at $d=d_3$ the potential energy has a local minimum, thus it is a stable position.

3. *Define a five-digit number* irreducible *if it cannot be expressed as a product of two three-digit numbers. What is the largest number of consecutive numbers that are all irreducible?* (1) 99; (2) 100; (X) *neither of these.*

Solution. For example, $10\,001, 10\,002, \dots, 10\,099$ are irreducible since they all lie between $100\cdot 100$ and $100\cdot 101$. On the other hand, in a sequence of 100 consecutive numbers one is always divisible by 100, that is not irreducible.

4. *Bubbles of air are rising in a test tube filled with oil. If an electrostatically charged glass rod is held next to the tube, it will* (1) *repel the bubbles;* (2) *attract the bubbles;* (X) *not affect the bubbles at all.*

Solution. The charged glass rod polarizes oil (a dielectric), and thus attracts it (independently of the sign of the charges). The air in the bubbles is also polarized to some extent and also experiences some attraction but that is negligible next to the force exerted on the oil. The effect of the horizontal (component of the) electric field directed towards the glass rod is similar to that of gravity: The bubbles will „rise", that is recede from the glass rod as if the glass rod were repelling them.

5. *Given that*

$$\frac{(a-b)(b-c)(c-a)}{(a+b)(b+c)(c+a)}=\frac{19}{99},$$

what is the value of $\frac{a}{a+b}+\frac{b}{b+c}+\frac{c}{c+a}$? (1) 1; (2) $\frac{101}{99}$; (X) $\frac{139}{99}$.

Solution. Let $x=a+b$, $y=b+c$, $z=c+a$. Then

$$\frac{19}{99}=\frac{(a-b)(b-c)(c-a)}{(a+b)(b+c)(c+a)}=\frac{(z-y)(x-z)(y-x)}{xyz}.$$

Since $a = \frac{1}{2}(x - y + z)$, $b = \frac{1}{2}(y - z + x)$, $c = \frac{1}{2}(z - x + y)$,

$$\frac{a}{a+b} + \frac{b}{b+c} + \frac{c}{c+a} = \frac{x-y+z}{2x} + \frac{y-z+x}{2y} + \frac{z-x+y}{2z}$$
$$= \frac{1}{2} - \frac{y-z}{2x} + \frac{1}{2} - \frac{z-x}{2y} + \frac{1}{2} - \frac{x-y}{2z}$$
$$= \frac{3}{2} - \frac{1}{2} \cdot \frac{(y-z)yz + (z-x)zx + (x-y)xy}{xyz}$$
$$= \frac{3}{2} - \frac{1}{2} \cdot \frac{(z-y)(x-z)(y-x)}{xyz} = \frac{3}{2} - \frac{1}{2} \cdot \frac{19}{99} = \frac{139}{99}.$$

6. *A strong bar magnet is falling in a vertical copper tube without touching the wall of the tube. If the weight of the tube is measured, it will be found* (1) *greater than without the magnet;* (2) *smaller than without the magnet;* (X) *the same as without the magnet.*

Solution. As a result of the retarding force of the eddy currents, the magnet will fall uniformly (disregarding a short time interval after starting). The momentum of the magnet is constant, and so is the total momentum of the tube and magnet system. Thus the resultant of the forces exerted on them (i.e. the gravitational forces acting on the two objects and the force exerted by the scales) must be zero. Therefore the scales will read the total „weight" of the tube and magnet even if the magnet does not touch the tube.

The same conclusions are reached by adding the reaction force of the upward retarding force acting on the magnet (equal in magnitude to its weight) to the weight of the tube (without the magnet).

7. *What is the digit in the tens' place in the smallest natural number that can be expressed as a sum of* 9 *consecutive positive integers as well as a sum of* 10 *consecutive positive integers?* (1) 0; (2) 2; (X) 3.

Solution. $x + (x+1) + \cdots + (x+8) = y + (y+1) + \cdots + (y+9)$, $9x + 36 = 10y + 45$, $0 < 9(x-1) = 10y$. Thus the smallest value of x is 11, and the number is 1**3**5.

8. *Assume that the atmosphere of the Earth has uniform temperature, and consider a vertical column of air that has a base area of* 1 m^2 *and reaches up to the „top" of the atmosphere. Which of the following is greater?* (1) *The internal energy of the gas.* (2) *The gravitational potential energy of the gas.* (X) *They are equal.*

Solution. If m denotes the mass of the column in question and H stands for the height of its centre of mass then the gravitational potential energy of the gas is $E_1 = mgH$ and its internal energy is $E_2 = \frac{5}{2}\frac{mRT}{M}$, where M is the molar mass of air and T is the temperature. The two energies would be equal if $H = 5RT/(2Mg)$ were true, for example, $T = 300$ K would need a value of $H \approx 20$ km. The density of the atmosphere decreases exponentially with height (if the temperature were constant, it would halve at 5-km intervals). That suggests a height of the centre of mass lying much lower than 20 km. (It can be shown by integration.) Therefore, the internal energy is greater than the gravitational potential energy (with respect to sea level).

9. *The distance between two skew edges of a regular tetrahedron is* 6 cm. *What is the volume of the tetrahedron in* cm^3? (1) 36; (2) 72; (X) 144.

Solution. Inscribe the tetrahedron in a cube so that each edge of the tetrahedron is a diagonal of a face of the cube. The edge of the cube is 6 cm and the volume of the tetrahedron is obtained from the volume of the cube by subtracting the total volume of the four „corners", that is $6^3 - 4 \cdot \frac{1}{6} \cdot 6^3 = 72$ cm^3.

10. *A cylindrical container is filled with water to half of its height. A ping pong ball is floating on the surface at a distance of* $\frac{1}{2}$ *radius from the centre. Will that distance change if the container is rotated about its axis of symmetry?* (1) *The ball will slide down the paraboloidal „slope".* (2) *The centrifugal force will „throw" the ball outwards.* (X) *The distance will not change since the water next to the ball does not slide downhill, nor is it thrown outwards.*

Solution. The ping pong ball will move closer to the axis of rotation but *not* because it is „sliding" down the sloping water surface (since the water is not sliding down either). The phenomenon is related to the fact that the centrifugal force in a rotating coordinate frame is nonuniform, i.e. depends on position. (For details, see the solution to Problem 1 of the 1990 Eötvös Competition in the 1991/2 issue of KöMaL.)

11. *Let* a, b, c, d, e, f *denote different integers. What is the smallest possible value of* $(a-b)^2 + (b-c)^2 + (c-d)^2 + (d-e)^2 + (e-f)^2 + (f-a)^2$? (1) 18; (2) 20; (X) 30.

Solution. The difference of the smallest and the largest of the six different numbers is at least 5. Thus the sum of a few consecutive numbers in the sequence $|a-b|, |b-c|, \ldots, |f-a|$ of differences is at least 5 and so is the sum of the remaining difference. As we move from the smallest number to the largest, directed differences add, just like along the other branch of the circuit. It follows from the inequality of the arithmetic and quadratic means that the sum in question is at least $16.\dot{6}$ that is 17. Since the sum is even, it must be at least 18, and that is indeed the case if the numbers are, for example, 0, 1, 3, 5, 4, 2.

12. *According to some theories, Planck's constant is not really a constant but increases very slowly with time. Assume that this is true, and the diffraction of (relatively slow) electron waves on a crystal is measured very accurately. Then one year later the measurement is repeated with all circumstances carefully reproduced. What will be the result of the repeated measurement?*

The angles of zero-order and first-order maxima will be: (1) *greater;* (2) *smaller;* (X) *the same as in the first measurement.*

Solution. In addition to Planck's constant (h), the diffraction angle (φ) of matter waves is determined by the speed of the particles (v), the mass of the electrons (m) and the crystal spacing (d): $\varphi \approx \sin\varphi = h/(mvd)$. It may seem that φ increases in proportion to h but it is not as simple as that! The size of the atoms and the spacing of atoms in the crystal also depend on Planck's constant: $d \sim h^2/(ke^2m)$ (where ke^2 is a constant characteristic of the Coulomb attraction of elementary charges and „$\sim$" stands for proportionality). Hence $\varphi \sim ke^2/(hv)$, which means that the angle φ decreases in inverse proportion to h, provided that v is constant. But is it?

The requirement of reproducing all circumstances means that the speed of the electrons (in m/s) stays constant. Since the metre and the second are defined so that the speed of light (c) in m/s has a certain (fixed) value, keeping the speed of the electrons constant in m/s means that their speed relative to the speed of light is constant, too. Hence the speed is a constant independent of h.

Putting it all together, we can state that the angle of diffraction is proportional to the dimensionless quantity $ke^2/(hc)$ (called the constant of fine structure), and thus it would decrease with a possible increase in h.

13. *Let n be a positive integer. Define the mapping $n \mapsto n'$ as follows: if p is a prime then $p' = 1$, and $(ab)' = a'b + b'a$ where a and b are natural numbers. How many two-digit numbers are there such that $n' = n$?* (1) 1. (2) 2. (X) *Neither of the two answers.*

Solution. If $n = p_1p_2 \dots p_r$ (p_i is a prime), then $n' = n\left(\frac{1}{p_1} + \frac{1}{p_2} + \dots + \frac{1}{p_r}\right)$. Thus $n = n'$ if and only if $\frac{1}{p_1} + \frac{1}{p_2} + \dots + \frac{1}{p_r} = 1$, which means that $p_1 = p_2 = \dots = p_r = p$, that is $n = p^p$ where p is a prime. The only such two-digit number is 27.

13+1. *Imagine an asteroid of radius R and uniform mass distribution made of pure gold. The acceleration of gravity is g_1 at a height h above the surface and g_2 at a depth h below the surface. Which is greater?* (1) $g_1 > g_2$. (2) $g_1 < g_2$. (X) *It depends on the value of the ratio h/R and the transition occurs at the golden ratio.*

Solution. At a height of $h = xR$ above the surface of the asteroid, the acceleration of gravity decreases according to $1/(x+1)^2$. Under the surface it is $(1-x)$ times the value at the surface. The two accelerations of gravity are equal if $\frac{1}{(x+1)^2} = 1 - x$, that is $x^2 + x - 1 = 0$. The solution of the equation (the one we are interested in) is $x = \frac{\sqrt{5}-1}{2} \approx 0.618$, which is equal to the golden ratio.